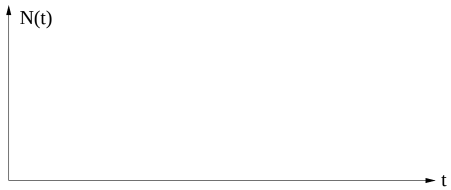
Renewal Processes/Counting Processes/Poisson Processes



$N(t)$: number of arrivals in $(0, t]$
S_n: Epoch of nth arrival
$\{X_1, X_2, X_3, ...\}$: interarrival intervals

$$X_1 = S_1, \quad X_n = S_n - S_{n-1}, \quad S_n = \sum_{i=1}^{n} X_i$$

Renewal Process: Interarrival intervals are positive IID random variables
(Renewal processes are more general than it might seem)

Counting Processes

$\{N(t), t \geq 0\}$: family of random variables
$N(t)$: number of arrivals in interval $(0, t]$.
$N(0) = 0$ with probability 1

Counting Process $\{N(t), t \geq 0\}$**:** family of non-negative integer valued random variables (one for each $t \geq 0$) with the properties that

$$N(\tau) \geq N(t), \qquad \tau \geq t$$

and $N(0) = 0$ with probability 1.

Equivalent: $\{S_1, S_2, ...\}$ or $\{X_1, X_2, ...\}$ or $\{N(t), t \geq 0\}$.

Note: $\{S_n \leq t\} = \{N(t) \geq n\}$

Poisson Process

- Renewal process with $F_X(x) = 1 - e^{-\lambda x}$
- λt: expected number of arrivals in interval of length t
- $E[N(t)] = \lambda t$
- Memoryless Property

$$P(X > t + x \mid X > t) = P(X > x), \qquad x \geq 0$$

- $P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

Theorem (informal statement):
(a) Interarrival interval from t until the first arrival after t is a R.V. with $F_X(x) = 1 - e^{-\lambda x}$
(b) This R.V. is independent of **all arrival epochs before time** t and $N(\tau)$ **for** $\tau \leq t$

Stationary Increment Property

$\{N(t), t \geq 0\}$: counting process

$\tilde{N}(t, t') = N(t') - N(t)$: number of arrivals in $(t, t']$, $t' \geq t$

$\tilde{N}(t, t')$ has same distribution as $N(t' - t)$

Independent Increment Property

$\{N(t_1), \tilde{N}(t_2, t_1), \tilde{N}(t_3, t_2)\}$ independent random variables

Definition Poisson Process

Definition 1: Renewal process with $F_X(x) = 1 - e^{-\lambda x}$

Definition 2: Counting process $\{N(t), t \geq 0\}$ with independent and stationary increment properties, and

- $P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

Definition 3: Counting process $\{N(t), t \geq 0\}$ with independent and stationary increment properties, and

- $\tilde{N}(t + \delta, t) = 0) = 1 - \lambda\delta + o(\delta)$
- $\tilde{N}(t + \delta, t) = 1) = \lambda\delta + o(\delta)$
- $\tilde{N}(t + \delta, t) \geq 2) = o(\delta)$

Combining Independent Poisson Processes

Bernoulli Splitting of Poisson Processes

The two processes are independent