Markov Processes

- $\{X(t); t \geq 0\}$: semi-Markov process ($P_{ii} = 0$)
- $P(U_n \leq x | X_{n-1} = i; X_n = j) = 1 - e^{-\nu_i x}, \quad \nu_i \geq 0$
- Markov process

  $$P(X(s) = j | X(t) = i, \{X(\tau) : \tau < t\}) = P(X(s) = j | X(t) = i), \quad s > t$$

- Visualization: 4 different ways

Markov Processes

- Assume that embedded M.C. is irreducible and positive recurrent:

  $$\pi_i = \sum_j \pi_j P_{ji}$$

- Assume $\sum_k \pi_k \frac{1}{\nu_k} < \infty$
- $p_i = \frac{\pi_i / \nu_i}{\sum_k \pi_k / \nu_k}$
- $\lim_{t \to \infty} P(X(t) = i) = p_i$
- $p_i \nu_i \sim \pi_i$
- $\pi_i = \frac{p_i \nu_i}{\sum_k p_k \nu_k}$
- Equilibrium Equation: $p_j \nu_j = \sum_i p_i P_{ij} \nu_i = \sum_i p_i q_{ij}$

Markov Processes

- **Irreducible** Markov process: the embedded M.C. is irreducible.
- **Theorem**: Assume a irreducible Markov process and let $\{p_i\}$ be a solution to

  $$p_j \nu_j = \sum_i p_i P_{ij} \nu_i = \sum_i p_i q_{ij}$$

  with $\sum_i p_i = 1$. If $\sum_i p_i \nu_i < \infty$, then, first, that solution is unique, second, for each i,

  $$p_i = \lim_{t \to \infty} P(X(t) = i),$$

  third, for each i, p_i is the time average fraction of time spent in state i w.p.1, and fourth, the embedded M.C. is positive recurrent.

Conversely, if the embedded M.C. is positive recurrent with steady state probabilities $\{\pi_i\}$, and $\sum_i \pi_i / \nu_i < \infty$, then the equation

$$p_j \nu_j = \sum_i p_i P_{ij} \nu_i = \sum_i p_i q_{ij}$$

has a unique solution with $\sum_i p_i = 1$ and for each i we have $p_i > 0$.

Markov Processes: Sampled Process

- Assume that $\{\nu_i\}$ are bounded
- Steady state probabilities $\{w_i\}$ for sampled process

$$w_j = \sum_{i \neq j} w_i q_{ij} \delta + w_j (1 - \nu_j \delta), \qquad w_j \geq 0,$$

such that $\sum_i w_i = 1$.

- $w_j \nu_j \delta = \sum_{i \neq j} w_i q_{ij} \delta$
- $w_j = p_j$!
- What about dynamics?

Markov Processes: Pathological Cases

Markov Processes: Uniformization

- Add arbitrary "self-transitions" with rate q_{ii}

$$\nu_i' = \sum_{j \neq i} q_{ij} \qquad \nu_i = \nu_i' + q_{ii}$$

  – Changes embedded M.C.:

$$P_{ij} = \frac{q_{ij}}{\nu_i} = \frac{q_{ij}}{\nu_i' + q_{ii}}$$

  – Does not change Markov process

$$p_i(\nu_i' + q_{ii}) = \sum_{i \neq j} p_i q_{ij} + p_i q_{ii}$$

Markov Processes: Uniformization

- Why useful?
- Choose q_{ii} such that

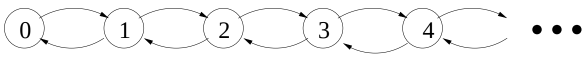
$$\nu_i = \nu_i' + q_{ii} = \nu^* = \sup_j \nu_j'$$

- For embedded M.C.

$$\pi_i = \frac{p_i \nu^*}{\sum_j p_j \nu^*} = p_i$$

- Useful for Markov Decision Processes

Birth Death Processes



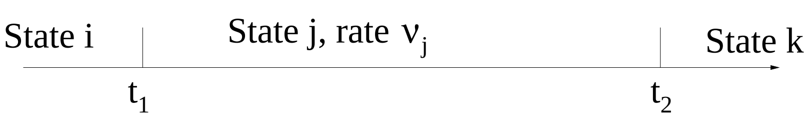
- $R_{i,i+1}(t)$: number of transition from state i to $i+1$ in $[0,t]$
- $R_{i+1,i}(t)$: number of transition from state $i+1$ to i in $[0,t]$
- We have $p_i\lambda_i = p_{i+1}\mu_{i+1}$

Markov Processes: Reversibility

- Embedded M.C.:

$$\pi_i P^*_{ij} = \pi_j P_{ji}$$

- Markov Process



- Steady state probabilities

$$p_i = \frac{\pi/\nu_i}{\sum_j \pi_j \nu_j}$$

Markov Processes: Reversibility

- Transition rates

$$q^*_{ij} = \nu_i P^*_{ij} = \frac{\nu_i \pi_j P_{ji}}{\pi_i} = \frac{\nu_i \pi_j q_{ji}}{\pi_i \nu_j}$$

or

$$p_i q^*_{ij} = p_j q_{ji}$$

- **Reversible** Markov process: $q^*_{ij} = q_{ij}$
- **Lemma:** Assume that the steady state probabilities $\{p_i\}$ exist in an irreducible Markov process and that $\sum_j p_j \nu_j < \infty$. Then the Markov process is reversible if and only if the embedded chain is reversible.

Markov Processes: Reversibility

- **Theorem**: For an irreducible Markov process, assume $\{p_i\}$ is a set of non-negative numbers summing to 1 such that $\sum_j p_j \nu_j < \infty$ and

$$p_i q_{ij} = p_j q_{ji}, \qquad \text{for all } i, j.$$

Then $\{p_j\}$ is the set of steady state probabilities for the process, $p_j > 0$ for all j, the process is reversible, and the embedded Markov chain is positive recurrent.

Markov Processes: Reversibility

- **Theorem**: For an irreducible Markov process, assume $\{p_i\}$ is a set of non-negative numbers summing to 1 such that $\sum_j p_j \nu_j < \infty$. Also assume that a set of non-negative numbers $\{q^*_{ij}\}$ satisfy the two set of equations

$$\sum_j q_{ij} = \sum_j q^*_{ij}, \qquad \text{for all } i$$

$$p_i q^*_{ij} = p_j q_{ji}, \qquad \text{for all } i, j.$$

Then $\{p_j\}$ is the set of steady state probabilities for the process, $p_j > 0$ for all j, the embedded Markov chain is positive recurrent, and the set $\{q^*_{ij}\}$ is the set of transition rates in the backward process

Markov Processes: Reversibility and Birth Death Process

- Birth death processes are reversible.
- $M/M/1$ queue