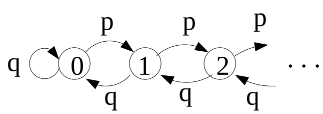
Markov Chains with Countably Infinite State Spaces



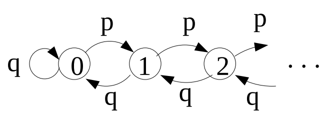
p = 1−q

- State 0: recurrent or transient?
- $p < q$
- $p = q$
- $p > q$

Renewal theory: Inter-arrival time must be a r.v.

Classification

- **recurrent**: state i is recurrent if $F_{ii}(\infty) = 1$
- **transient**: state i is transient if $F_{ii}(\infty) < 1$
- Note that above definition are consistent with the ones for finite state M.C. (what about the other way around?)



p = 1−q

Classification

If $F_{ij}(\infty) = 1$, then the mean time $\bar{T}_{ij}$ is of interest,

$$\bar{T}_{ij} = 1 + \sum_{n=1}^{\infty} \Big(1 - F_{ij}(n)\Big).$$

$\bar{T}_{ij}$ can be finite or infinite.

- **null recurrent**: state i is null recurrent if $F_{ii}(\infty) = 1$ and $\bar{T}_{ii} = \infty$.
- **positive recurrent**: state i is positive recurrent if $F_{ii}(\infty) = 1$ and $\bar{T}_{ii} < \infty$.
- **transient**: state i is transient if $F_{ii}(\infty) < 1$

Renewal Theory

Assume that state j is recurrent and consider the renewal process $\{N_{jj}(t); t \geq 0\}$. Then

- $\lim_{t\to\infty} \frac{N_{jj}(t)}{t} = \frac{1}{E[T_{jj}]}, \qquad w.p.1$
- $\lim_{t\to\infty} \frac{E\left[N_{jj}(t)\right]}{t} = \frac{1}{E[T_{jj}]}$
- If state j is periodic with span d, then

$$\lim_{n\to\infty} P(X_{nd} = j \mid X_0 = j) = \frac{d}{\bar{T}_{jj}}$$

- If state j is aperiodic, then

$$\lim_{n\to\infty} P(X_n = j \mid X_0 = j) = \frac{1}{\bar{T}_{jj}}$$

Steady-State probabilities

Theorem: Let j be a **recurrent** state in a M.C. and let i be any state **in the same class**. Furthermore, let $X_0 = i$ and consider the counting process $\{N_{ij}(t); t \geq 0\}$. Then

- $\lim_{t\to\infty} \frac{N_{ij}(t)}{t} = \frac{1}{E[T_{jj}]}, \qquad w.p.1$
- $\lim_{t\to\infty} \frac{E\left[N_{ij}(t)\right]}{t} = \frac{1}{E[T_{jj}]}$
- If state j is aperiodic, then

$$\lim_{n\to\infty} P(X_n = j \mid X_0 = i) = \frac{1}{\bar{T}_{jj}}$$

Theorem: All states in the same class of a M.C. are of the same type - positive recurrent, null recurrent, transient.

Steady-State Distribution

Vector $(\pi_0, \pi_1, \pi_2, ...)$, $\pi_i \geq 0$, such that

$$\pi_i = \sum_j \pi_j P_{ji}$$

and

$$\sum_i \pi_i = 1$$

- **Theorem:** Consider an irreducible M.C. with transition probabilities $\{P_{ij}\}$. If the above equation has a solution, the the solution is **unique**, we have $\pi_i = \frac{1}{T_{ii}}$ for all $i \geq 0$, and all states are **positive recurrent**. Also, if all states are positive recurrent, then the above equation has a solution.