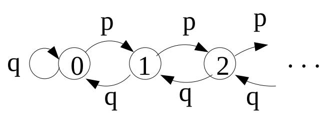
Markov Chains with Countably Infinite State Spaces



p = 1−q

- State 0: recurrent or transient?
- $p < q$
- $p = q$
- $p > q$

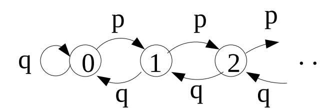
Renewal theory: Interarrival time must be a r.v.

First Passage-Time

- $f_{ij}(n)$: "first passage time probability"
  - $f_{ij}(n) = P\Big(X_n = j, X_{n-1} \neq j, ..., X_1 \neq j | X_0 = i\Big)$
  - $P_{ij}(X_n = j | X_0 = i) \qquad f_{ij}(n)$
- $f_{ij}(n) = \sum_{k \neq j} P_{ik} f_{kj}(n-1), \qquad n > 1$ and $f_{ij}(1) = P_{ij}$
  We can compute $f_{ij}(n)$ recursively
- $F_{ij}(n) = \sum_{m=1}^{n} f_{ij}(m)$
- T_{ij}: first passage time
  - $f_{ij}(n)$: probability mass function
  - $F_{ij}(n)$: probability distribution functionq
- $F_{ij}(\infty)$: If $F_{ij}(\infty) = 1$ then T_{ij} is a R.V. - otherwise T_{ij} is a defective R.V.

Classification

- **recurrent**: state i is recurrent if $F_{ii}(\infty) = 1$
- **transient**: state i is transient if $F_{ii}(\infty) < 1$
- Note that above definition are consistent with the ones for finite state M.C. (what about the other way around?)



p = 1−q

$F_{ij}(\infty)$

- $F_{ij}(n) = P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(n-1); \qquad n > 1, \quad F_{ij}(1) = P_{ij}$
- $F_{ij}(n)$: non-decreasing in n and upper-bounded by 1 - therefore $\lim_{n \to \infty} F_{ij}(n)$ exists.
- $F_{ij}(\infty) = P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(\infty)$
  - does a solution exist?
  - is there a unique solution?
- **Lemma:** Let state i be accessible from j and let j be recurrent. Then $F_{ij}(\infty) = 1$

Renewal Theory

- Given $X_0 = i$, let $\{N_{ij}(t); t \geq 0\}$ be the counting process where $N_{ij}(t)$ is the number of transitions into state j by time t (including self-transitions).
- Similarly, $\{N_{jj}(t); t \geq 0\}$
- If j is **recurrent**, then $\{N_{jj}(t); t \geq 0\}$ is a renewal process.
- We know (whether or not $E[T_{jj}]$ is finite)
  - $\lim_{t\to\infty} N_{jj}(t) = \infty$, w.p. 1
  - $\lim_{t\to\infty} E[N_{jj}(t)] = \infty$
- If j is **transient**, then $\{N_{jj}(t); t \geq 0\}$ is **not** a renewal process.
  - Expected number of returns: $\frac{F_{jj}(\infty)}{1-F_{jj}(\infty)} < \infty$

Renewal Theory

- $P_{jj}^n = E\Big[N_{jj}(n) - N_{jj}(n-1)\Big]$
- $E\Big[N_{jj}(n)\Big] = \sum_{k=1}^n P_{jj}^k$

Results

- **Lemma**: Let $\{N_{jj}(t); t \geq 0\}$ be the counting process for occurrences of state j up to time t in a Markov chain starting at state j. The state j is recurrent if and only if

$$\lim_{t\to\infty} N_{jj}(t) = \infty, \qquad \text{w.p.1}$$

  Also, j is recurrent if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{n} P_{jj}^k = \infty.$$

- **Lemma**: If state i is recurrent and state i and j are in the same class, the state j is recurrent.

Results

- **Lemma**: Let $\{N_{ij}(t); t \geq 0\}$ be the counting process for occurrences of state j up to time t in a Markov chain starting at state i. Then if i and j are **in the same class** and i and j are **recurrent**, then $\{N_{ij}(t); t \geq 0\}$ is a delayed renewal process.