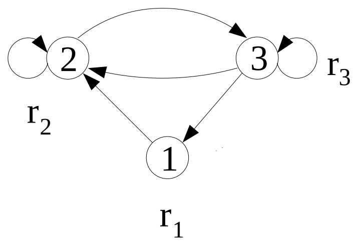
Finite State Markov Chain with Rewards

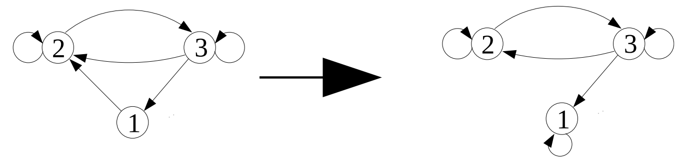


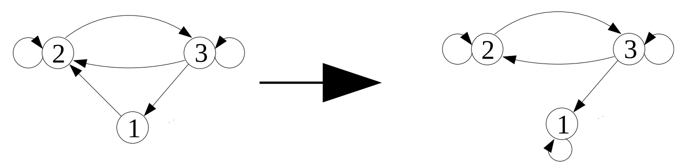
- $S = \{1, ..., J\}$
- Reward r_i in state $i \in S$
- $\{X_n; n \geq 0\}$
- $\{R_n; n \geq 0\}$

- $v_i(n) = E\Big[\sum_{k=0}^{n} R_k \mid X_o = i]$
- $g = \lim_{n\to\infty} \frac{1}{n} E\Big[\sum_{k=0}^{n-1} R_k \mid X_o = i]$
- Single Recurrent Class: $g = \sum_{i=1}^{J} \pi_i r_i$
- Focus on $v_i(n)$

Example: First Passage Time

- v_2: "expected number of steps to reach state 1 from state 2"
- Approach: Introduce "trapping" state





- Assume that we know v_2, v_3; and set $v_1 = 0$. Then

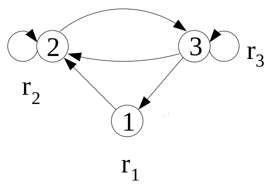
$$v_2 = 1 + P_{22}v_2 + P_{23}v_3$$
$$v_3 = 1 + P_{32}v_2 + P_{33}v_3$$

or

$$v = r + [P]v, \qquad r_i = \begin{cases} 1, & i \neq 1 \\ 0, & \text{otherwise} \end{cases}$$

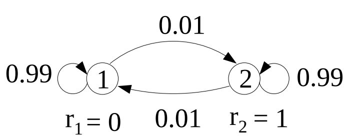
- $v_i = \lim_{n\to\infty} v_i(n)$
- Does there exist a solution? Unique Solution?

Example: Markov Chain without Trapping State



- $[P], r, \pi = \pi[P]$
- Suppose that $g = \sum_{i=1}^{J} \pi_i r_i = 0$
- $v_i = \lim_{n\to\infty} v_i(n)$
- $v = r + [P]v$
- Does there exist a solution?
- Is the solution unique?

Example: Relative Gain



- $\lim_{n\to\infty} \frac{1}{n} E\Big[\sum_{k=1}^{n} R_k \mid X_o = i\Big] = g = \sum_{i=1}^{J} \pi_i r_i$
- Time-average is the same no matter where we start.
- What about transient behavior: $v_1(n) - v_2(n)$?

Review: Linear Algebra

- $Ax = b$, A is a $J \times J$ matrix
- $rank(A)$: number of linearly independent rows/columns
- $rank(A, b)$: number of linearly independent equations
  - if $rank(A, b) \neq rank(A)$: no solution
  - if $rank(A, b) = rank(A) = J$: unique solution
  - if $rank(A, b) = rank(A) = \rho < J$: $(J - \rho)$-dimensional set of solutions

Conjecture for $v(n)$

- Observations
  - $v(n) = r + [P]v(n-1) = r + [P]r + ... + [P]^{n-1}r + [P]^n v(0)$.
  - $v(n) - nge =$
    $(r - ge) + [P](r - ge) + ... + [P]^{n-1}(r - ge) + [P]^n v(0)$
- Conjecture: $\lim_{n\to\infty} v(n) - nge = w + \pi v(0)e$,

$$\pi w = 0$$

- Observation

$$\lim_{n\to\infty} (v(n) - nge) = \lim_{n\to\infty} (r + [P]v(n-1) - nge).$$

or

$$w + ge = [P]w + r$$

Solution

- Does there exists a unique solution for

$$w + ge = [P]w + r$$

with

$$\pi w = 0$$

- Obervation: $([P] - I)$ has rank $J - 1$
- Question: Does $([P] - I, ge - r)$ have rank $J - 1$?

Finite State Markov Chain with Rewards

- Single Recurrent Class: $v(n) = nge + w + [P]^n\{v(0) - w\}$
- Ergodic: $\lim_{n\to\infty}\{v(n) - nge\} = w + \beta e;, \qquad \beta = \pi(v(0) - w)$