Finite State Markov Chain

- $S = \{1, ..., J\}$
- Matrix Representation

$$[P] = \begin{bmatrix} P_{11} & \cdots & P_{1J} \\ \vdots & & \vdots \\ P_{J1} & \cdots & P_{JJ} \end{bmatrix}$$

- Classification:
  - recurrent / transient
  - periodic / aperiodic
- ergodic class
- ergodic chain: single ergodic class

Finite State Markov Chain

- **Theorem**: For ergodic Markov chain we have

$$P_{ij}^m > 0, \qquad m \geq J(J-1).$$

- Questions
  - Does $[P]^n$ converge?
  - Is $lim_{n\to\infty} P_{ij}^n = lim_{n\to\infty} P_{kj}^n$?
  - If above is true, then $\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$

    and $\pi = \pi[P]$

Finite State Markov Chain

- Questions
  - Does $\pi = \pi[P]$ have a probability vector solution?
  - Does $\pi = \pi[P]$ have a unique probability vector solution?
  - Is $\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$

Answers

- Yes, solution to $\pi = \pi[P]$ always exists
- Unique, if and only if there is a single recurrent class (and possibly many transient classes)
- If there are r recurrent classes, then there exist r linearly independent solutions.
- For ergodic Markov chain we have

  $\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$
- If there are several multiple ergodic classes $\lim_{n\to\infty}[P]^n$ exists, but rows are not identical.
- If there is one or more periodic class then $[P]^n$ does not converge.

Answers

- Existence: yes
- Uniqueness: single recurrent class
- Convergence: aperiodic
- Uniqueness and Convergence: ergodic

Matrix Theory

- $[A]$: $J \times J$ matrix
- π is **left eigenvector** with **eigenvalue** λ if $\pi \neq 0$ and

$$\pi[A] = \lambda\pi$$

- ν is **right eigenvector** with **eigenvalue** λ if $\pi \neq 0$ and

$$[A]\nu = \lambda\nu$$

- Eigenvectors with eigenvalue λ exist if $[A - \lambda I]$ is singular, *i.e.*

$$\det([A - \lambda I]) = 0$$

Example

- $[P] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$
- Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 1 - P_{12} - P_{21}$
- Eigenvectors: $\pi^{(1)}, \pi^{(2)}$ and $\nu^{(1)}, \nu^{(2)}$
- $[\Lambda] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$; $[U] = \begin{bmatrix} \nu_1^{(1)} & \nu_1^{(2)} \\ \nu_2^{(1)} & \nu_2^{(2)} \end{bmatrix}$;
- $[P][U] = [U][\Lambda]$; $[U]^{-1}[P] = [\Lambda][U]^{-1}$; $[U]^{-1} = \begin{bmatrix} \pi_1^{(1)} & \pi_2^{(1)} \\ \pi_1^{(2)} & \pi_2^{(2)} \end{bmatrix}$;

Example

- $[P] = [U][\Lambda][U]^{-1}$; $\qquad [P]^n = [U][\Lambda]^n[U]^{-1}$
- $[P]^n = \begin{bmatrix} \pi_1 + \pi_2\lambda_2^n & \pi_2 - \pi_2\lambda_2^n \\ \pi_1 - \pi_1\lambda_2^n & \pi_2 + \pi_1\lambda_2^n \end{bmatrix}$; $\begin{bmatrix} \pi_1 = \frac{P_{21}}{P_{12}+P_{21}} = \pi_1^{(1)} \\ \pi_2 = 1 - \pi_1 = \pi_2^{(1)} \end{bmatrix}$;
- $\lambda_2 = 1 - P_{12} - P_{21}$ and $\mid \lambda_2 \mid \leq 1 = \lambda_1$
- If $P_{12} = P_{21} = 0$ then $\lambda_2 = 1$ and $[P] = [I]$ (two recurrent classes)
- If $P_{12} = P_{21} = 1$ then $\lambda_2 = -1$ and $[P]^n = [P]$ for n odd, and $[P]^n = [I]$ for n even (periodic)
- In all other cases we have $\mid \lambda_2 \mid < 1$ and $\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}$; (ergodic = aperiodic,recurrent)

Questions

- Does there exists a left eigenvector π with eigenvalue $\lambda = 1$, *i.e.*

$$\pi = \pi[P]$$

- Does $\lim_{n \to \infty} [P]^n$ converge?

Perron-Frobenius Theory

- $x = (x_1, ...x_J)$ is **positive** ($x > 0$) if $x_j > 0$, $j = 1, ..., J$
- $[A] = [A_{ij}]$ is **positive** ($[A] > 0$) if $A_{ij} > 0$, $i, j = 1, ..., J$
- $x = (x_1, ...x_J)$ is **non-negative**: $x \geq 0$
- $[A]$ is **non-negative**: $[A] \geq 0$
- $x > y$ if $(x - y) > 0$
- A $J \times J$ matrix $[A]$ is **irreducible** if $[A] \geq 0$ and $\sum_{n=1}^{J-1} [A]^n > 0$ (single recurrent class)

Theorem (Frobenius)

- $[A]$, $J \times J$ matrix which is **irreducible**, then
  - $[A]$ has a **positive** eigenvalue λ that is greater than or equal to the magnitude of every other eigenvalue, *i.e.* $\lambda \geq |\lambda'|$
  - There is a **positive right eigenvector** $\nu > 0$ **corresponding to** λ such that
    - for any non-zero vector $x \geq 0$, if $\lambda x \leq [A]x$, then $\lambda x = [A]x$
    - if $\lambda x = [A]x$ then $x = \alpha\nu$ ("ν is unique within a scale factor")

Corollaries

- **Corollary 1:** The **largest real eigenvalue** of an irreducible matrix $[A] \geq 0$ has a **positive left eigenvector** π. π is unique (within a scale factor) and it is the **only non-negative, non-zero vector** (within a scale factor) that satisfies $\lambda\pi \leq \pi[A]$

- **Corollary 2:** Let λ be the **largest real eigenvalue** of an irreducible matrix $[A]$ and let $\pi > 0$, $\nu > 0$, be the eigenvectors. Then (within a scale factor) ν and π are the only non-negative eigenvectors.

Corollaries

- **Corollary 3:** Let P be a **stochastic** irreducible matrix $[A]$, then
  - $\lambda = 1$ is the largest real eigenvalue of $[P]$
  - $e = (1, ..., 1)^T$ is the right eigenvector of $\lambda = 1$
  - there is a unique probability vector $\pi > 0$ that is a left eigenvector for $\lambda = 1$

  Extension: stochastic matrix $[P]$ always has a probability vector $\pi > 0$ that is a left eigenvector for $\lambda = 1$
- **Corollary 4:** Let P be the transition matrix of a Markov chain with a single recurrent class and one or more transient classes, then
  - $\lambda = 1$ is the largest real eigenvalue of $[P]$
  - $e = (1, ..., 1)^T$ is the right eigenvector of $\lambda = 1$
  - there is a unique probability vector $\pi \geq 0$ that is a left eigenvector for $\lambda = 1$

Corollaries

- **Corollary 6:** Let λ be the largest eigenvalue of a **positive matrix** $[A] > 0$ and let π (ν) be the positive left (right) eigenvector of λ normalized such that $\pi\nu = 1$, then $\lim_{n\to\infty} \frac{[A]^n}{\lambda^n} = \nu\pi$
- **Theorem 6:** Let $[P]$ be the transition matrix of an *ergodic* Markov chain. Then $\lambda = 1$ is the largest real eigenvalue of of $[P]$, and $\lambda \geq |\lambda'|$ for every other eigenvalue λ'. Furthermore, we have $\lim_{n\to\infty} \frac{[P]^n}{\lambda^n} = e\pi = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$
- Theorem 6 can be extended to Markov chains with a single ergodic class and transients.

Review Answers

- Existence: yes - Extension Corollary 3/4
- Uniqueness: single recurrent class - Corollary 3/4
- Convergence: aperiodic - Extension Theorem 6
- Uniqueness and Convergence: ergodic - Theorem 6