System Modelling - Markov Chain

- Example: Systems is either "idle" or "busy"
- Reward r_i in state $i \in \{0, 1\}$
- System Dynamic: $X_0, X_1, X_2, X_3, ...,\qquad X_n \in \{0, 1\}$
  X_0: fixed, or choose with some probability
- Rewards: $R_0, R_1, R_2, R_3, ...,\qquad R_n \in \{r_0, r_1\}$

- Questions:

$$P(X_n = j), \quad j \in \{0, 1\}, n = 0, 1, 2, ...$$
$$lim_{n\to\infty} P(X_n = j)$$
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} R_n$$
$$\lim_{n\to\infty} E[R_n]$$

Markov Chain

- Integer time process: $X_0, X_1, X_2, X_3, ...,\qquad X_n \in S$, such that

$$\begin{aligned} &P(X_n = j \mid X_{n-1} = i, X_{n-2} = k, ..., X_0 = m\} \\ &\quad = P(X_n = j \mid X_{n-1} = i) \\ &\quad = P_{ij} \end{aligned}$$

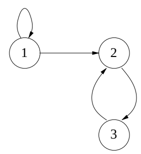
- P_{ij}: "transition probability for going from state i to j"
  (does **not** depend on time n)
- Nice: "independence", "decoupling", from the past $->$ reduction of complexity
- Initial distribution: $P(X_0 = i), \qquad i \in S$
- State space S: finite (matrix representation)
  countable infinite (renewal process)

Finite State Markov Chain

- $S = \{1, ..., J\}$
- State Transition diagram
- Matrix Representation

$$[P] = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$$

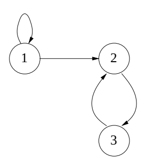
Classification of States



- **“j is accessible from i”**: there exists a directed path from i to j
  $i \to j$

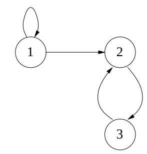
$$P(X_n = i \mid X_0 = i) > 0, \quad \text{for some } n \geq 1$$
$$P(X_n = i \mid X_0 = i) = P_{ij}^n, \qquad n \geq 1$$
$$P_{jj}^0 = 1; \qquad P_{ij}^0 = 0, \quad i \neq j$$

Classification of States



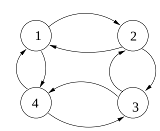
- **“i and j communicate”**:
  $i \leftrightarrow j$ if $i \to j$ and $j \to i$
  if $i \leftrightarrow m$ and $m \leftrightarrow j$, then $i \leftrightarrow j$
  $i \to i$
- **Class of states:** a **non-empty** set of states T such that
  - all states in T communicate with each other
  - no state in T communicates with a state that is not in T

Classification of States



- **“j is recurrent”**: if j is accessible from all states that are accessible from j
  if $j \to i$ then $i \to j$
- **“if j is not recurrent, then state j is transient”**
  - “eventually we will not observe j anymore”
  - if j is transient, then all states in the same class as j are transient
  - a class T is either recurrent, or transient
  - one “rotten state” makes the whole class “rotten”

Aperiodic/Periodic States



- if $X_0 \in \{2, 4\}$, then
  - $X_1 \in \{1, 3\}$
  - $X_2 \in \{2, 4\}$
  - $X_3 \in \{1, 3\}$
  - $X_4 \in \{2, 4\}$
- **Period** $d(i)$ of state i is the greatest common divisor (gcd) of the values n for which $P_{ii}^n > 0$

Aperiodic/Periodic States

- if $d(i) = 1$, then state i is **aperiodic**
- if $d(i) > 1$, then state i is **periodic**
- periodic vs. arithmetic
- **Theorem**: For any MC, all states of the same class have the same period.
- **Theorem**: If a recurrent class of states in a MC has period $d > 1$, then the class can be partitioned into d subclasses, $T_1, T_2, ..., T_d$, such that for each m, if $j \in T_m$ and $P_{ik} > 0$ then $k \in T_{m+1}$.

Ergodic Class

- **Ergodic Class**: both **recurrent** and **aperiodic**
- **Ergodic Chain**: consisting entirely of a single ergodic class
- **Theorem**: If a finite state MC is ergodic and has J states, then $P_{ij}^m > 0$ for all i, j and all $m \geq J(J-1)$.