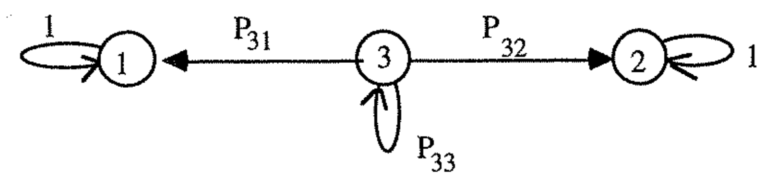
4.21) a) Find $\lim_{n\to\infty}[P]^n$ for the Markov chain below. Hint: Think in terms of the long term transition probabilities. Recall that the edges in the graph for a Markov chain correspond to the positive transition probabilities.

b) Let $\pi^{(1)}$ and $\pi^{(2)}$ denote the first two rows of $\lim_{n\to\infty}[P]^n$ and let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ denote the first two columns of $\lim_{n\to\infty}[P]^n$. Show that $\pi^{(1)}$ and $\pi^{(2)}$ are independent left eigenvectors of [P], and that $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are independent right eigenvectors of [P]. Find the eigenvalue for each eigenvector.



c) Let $\mathbf{r}$ be an arbitrary reward vector and consider the equation

$$[P]\mathbf{w} + \mathbf{r} = \mathbf{g} + \mathbf{w};\ \mathbf{g} = \alpha\mathbf{u}^{(1)} + \beta\mathbf{u}^{(2)};\ \alpha,\beta \text{ scalars} \qquad \text{(a)}$$

Determine what values α and β (and thus $\mathbf{g}$) must have in order for (a) to have a solution. Argue that with the additional constraints $w_1=w_2=0$, (a) has a unique solution for $\mathbf{w}$ and find that $\mathbf{w}$.

d) Show that $\mathbf{w}' = \mathbf{w} + \alpha\mathbf{u}_1 + \beta\mathbf{u}_2$ satisfies (a) for all choices of scalars α and β.

e) Assume that the reward at stage 0 is $\mathbf{v}(0) = \mathbf{w}$. Show that $\mathbf{v}(n) = n\mathbf{g} + \mathbf{w}$.

f) For an arbitrary reward $\mathbf{v}(0)$ at stage 0, show that $\mathbf{v}(n) = n\mathbf{g} + \mathbf{w} + [P]^n(\mathbf{v}(0)\text{-}\mathbf{w})$. Why isn't theorem 7 applicable here?

4.22) Generalize exercise 4.21 to the general case of two ergodic Markov classes and one transient class.

4.23) a) Consider a Markov decision problem with J states and assume that for each policy $\mathbf{A} = (k_1, k_2, ..., k_J)$, the Markov chain $[P^A]$ is ergodic. Suppose that policy **B** is a stationary optimal policy and **A** is any other policy. Use equations (50) to (52), with the inequalities appropriately reversed, to show that $g^B \geq g^A$.

b) Show that if $\mathbf{r}^B + [P^B]\mathbf{w}^B \geq \mathbf{r}^A + [P^A]\mathbf{w}^B$ is not satisfied with equality, then $g^B > g^A$.

c) Assume in parts (c) through (g) that **A** and **B** are both optimal stationary policies. Show that $\mathbf{r}^B + [P^B]\mathbf{w}^B = \mathbf{r}^A + [P^A]\mathbf{w}^B$. Hint: use part (b).

d) Find the relationship between the relative gain vector $\mathbf{w}^A$ for policy **A** and the relative gain vector $\mathbf{w}^B$ for policy **B**. Hint: Show that $\mathbf{r}^B + [P^B]\mathbf{w}^A = g^B\mathbf{e}+\mathbf{w}^A$; what does this say about $\mathbf{w}^A$?

e) Suppose that policy **A** uses decision 1 in state 1 and policy **B** uses decision 2 in state 1 (i.e., $k_1 = 1$ for policy **A** and $k_1 = 2$ for policy **B**). What is the relationship between $r_1^k, P_{11}^k, P_{12}^k, \ldots P_{1J}^k$ for k equal to 1 and 2?

f) Now suppose that policy **A** uses decision 1 in each state and policy **B** uses decision 2 in each state. Is it possible that $r_i^1 > r_i^2$ for all i? Explain carefully.

g) Now assume that r_i^1 is the same for all i. Does this change your answer to part (f)? Explain.

4.24) Consider a Markov decision problem with three states. Assume that each stationary policy corresponds to an ergodic Markov chain. It is known that a particular policy $\mathbf{B} = (k_1, k_2, k_3) = (2,4,1)$ is the unique optimal stationary policy (i.e., the gain per stage in steady state is maximized by always using decision 2 in state 1, decision 4 in state 2, and decision 1 in state 3). As usual, r_i^k denotes the reward in state i under decision k, and P_{ij}^k denotes the probability of a transition to state j given state i and given the use of decision k in state i.

Consider the effect of changing the Markov decision problem in each of the following ways (the changes in each part are to be considered in the absence of the changes in the other parts):

a) r_1^1 is replaced by $r_1^1 - 1$.

b) r_1^2 is replaced by $r_1^2 + 1$.

c) r_1^k is replaced by $r_1^k + 1$ for all state 1 decisions k.

d) for all i, $r_i^{k_i}$ is replaced by $r_i^{k_i} + 1$ for the decision k_i of policy **B**.

For each of the above changes, answer the following questions; *give explanations:*

i) Is the gain per stage, g^B, increased, decreased, or unchanged by the given change?
ii) Is it possible that another policy, $\mathbf{A}\neq\mathbf{B}$, is optimal after the given change?

4.25) (The Odoni Bound) Let **B** be the optimal stationary policy for a Markov decision problem and let g^B and π^B be the corresponding gain and steady state probability respectively. Let $v_i^*(n)$ be the optimal dynamic expected reward for starting in state i at stage n.

a) Show that $\min_i [v_i^*(n)\text{-}v_i^*(n\text{-}1)] \leq g^B \leq \max_i[v_i^*(n)\text{-}v_i^*(n\text{-}1)]$; $n\geq 1$. Hint: Consider premultiplying $\mathbf{v}^*(n) - \mathbf{v}^*(n\text{-}1)$ by π^B or π^A where **A** is the optimal dynamic policy at stage n.

b) Show that the lower bound is non-decreasing in n and the upper bound is non-increasing in n and both converge to g^B with increasing n.

4.26) (Extension of theorem 8 to recurrent plus transient chains). Define a policy **B** that is recurrent plus transient to be a stationary optimal policy if, first, $g^B \geq g^A$ for any other policy (or any recurrent class of any other policy with multiple recurrent classes), and second, if $g^B = g^A$, $\mathbf{w}^B \geq \mathbf{w}^A$, where $w_1=0$ for both policy **A** and **B** and state 1 is in the recurrent class of policy **B**.

a) Show that if (46) is satisfied and if $\mathbf{v}(0) = \mathbf{w}^B$, then the optimal dynamic policy is **B** at every stage.

behavior. The somewhat startling result here is that in steady state, and at a fixed time, the number of customers at each node is independent of the number at each other node and satisfies the same distribution as for an M/M/1 queue. Also the exogenous departures from the network are Poisson and independent from node to node. We emphasized that the number of customers at one node at one time is often dependent on the number at other nodes at other times. The independence holds only when all nodes are viewed at the same time.

For further reading on Markov processes, see [Kel79], [Ros83], [Wol89], and [Fel66].

EXERCISES

6.1) Consider a Markov process for which the embedded Markov chain is a birth death chain with transition probabilities $P_{i,i+1} = 2/5$ for all $i \geq 1$, $P_{i,i-1} = 3/5$ for all $i \geq 1$, $P_{01} = 1$, and $P_{ij} = 0$ otherwise.

a) Find the steady state probabilities $\{\pi_i;\ i \geq 0\}$ for the embedded chain.

b) Assume that the transition rate ν_i out of state i, for $i \geq 0$, is given by $\nu_i = 2^i$. Find the transition rates $\{q_{ij}\}$ between states and find the steady state probabilities $\{p_i\}$ for the Markov process. Explain heuristically why $\pi_i \neq p_i$.

c) Now assume in parts (c) to (f) that the transition rate out of state i, for $i \geq 0$, is given by $\nu_i = 2^{-i}$. Find the transition rates $\{q_{ij}\}$ between states and draw the directed graph with these transition rates.

d) Show that there is no probability vector solution $\{p_i;\ i \geq 0\}$ to Eq. 9 in Chapter 6.

e) Argue that the expected *time* between visits to any given state i is infinite. Find the expected *number of transitions* between visits to any given state i. Argue that, starting from any state i, an eventual return to state i occurs with probability 1.

f) Consider the sampled time approximation of this process with $\delta = 1$. Draw the graph of the resulting Markov chain and argue why it must be null-recurrent.

6.2) a) Consider a Markov process with the set of states $\{0, 1, \ldots\}$ in which the transition rates $\{q_{ij}\}$ between states are given by $q_{i,i+1} = (3/5)2^i$ for $i \geq 0$, $q_{i,i-1} = (2/5)2^i$ for $i \geq 1$, and $q_{ij} = 0$ otherwise. Find the transition rate ν_i out of state i for each $i \geq 0$ and find the transition probabilities $\{P_{ij}\}$ for the embedded Markov chain.

b) Find a solution $\{p_i\}$ with $\sum_i p_i = 1$ to (9).

c) Show that all states of the embedded Markov chain are transient.

d) For each state i of the embedded Markov chain, associate a reward $r_i = 2^{-i}$, i.e., the mean time until a transition is made from state i. Let $V_i(0)$ be a final reward in stage 0 and assume that $V_i(0) = W_i$ where W_i satisfies the equations

$$W_i = 2^{-i} + (3/5)W_{i+1} + (2/5)W_{i-1};\ i \geq 1 \text{ and } W_0 = 1 + W_1 \qquad \text{(a)}$$

Show that for all $n > 0$, $V_i(n) = W_i$. Note that this is the expected time to make n transitions, plus the final reward after n transitions. Thus if (a) has a bounded solution, the expected time to make an infinite number of transitions is finite, and the process is irregular.

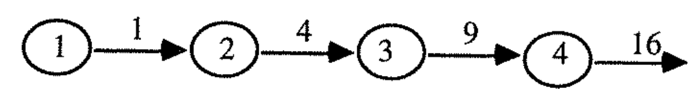
e) Define $\delta_i = W_i - W_{i+1}$, and show that (a) can be rewritten as $\delta_i = (2/3)\delta_{i-1} + (5/3)2^{-i}$ for $i \geq 1$. Note from (a) that $\delta_0 = 1$. Show that $\delta_i \leq 2i\,(2/3)^{i-1}$ for $i \geq 1$.

f) Show that $W_0 - \lim_{i\to\infty} W_i \leq \infty$. This shows that the process is irregular, and in particular shows that the solution $\{p_i;\ i \geq 0\}$ found in (a) is not a steady state solution, and in fact has no physical meaning.

6.3) a) Consider the process in the figure below. The process starts at $X(0) = 1$, and for all $i \geq 1$, $P_{i,i+1} = 1$ and $\nu_i = i^2$ for all i. Let T_n be the time that the n^{th} transition occurs. Show that

$$E[T_n] = \sum_{i=1}^{n} i^{-2} < 2 \quad \text{for all n}$$

Hint: Upper bound the sum from i=2 by integrating x^{-2} from x=1.



b) Use the Markov inequality to show that $P(T_n > 4) \leq 1/2$ for all n. Show that the probability of an infinite number of transitions by time 4 is at least 1/2.

6.4) Let $q_{i,i+1} = 2^{i-1}$ for all $i \geq 0$ and let $q_{i,i-1} = 2^{i-1}$ for all $i \geq 1$. All other transition rates are 0.

a) Solve the steady state equations and show that $p_i = 2^{-i-1}$ for all $i \geq 0$.

b) Find the transition probabilities for the embedded Markov chain and show that the chain is null recurrent.

c) For any state i, consider the renewal process for which the Markov process starts in state i and renewals occur on each transition to state i. Show that, for each $i \geq 1$, the expected inter-renewal interval is equal to 2. Hint: Use renewal reward theory.

d) Show that the expected number of transitions between each entry into state i is infinite. Explain why this does *not* mean that an infinite number of transitions can occur in a finite time.

6.5) A two state Markov process has transition rates $q_{01} = 1$, $q_{10} = 2$. Find $P_{01}(t)$, the probability that $X(t) = 1$ given that $X(0) = 0$. Hint: You can do this by solving a single first order differential equation if you make the right choice between forward and backward equations.

6.6) a) Consider a two state Markov process with $q_{01} = \lambda$ and $q_{10} = \mu$. Find the eigenvalues and eigenvectors of the transition rate matrix [Q].

b) Use (25) to solve for [P(t)].

c) Use the Kolmogorov forward equation for $P_{01}(t)$ directly to find $P_{01}(t)$ for $t \geq 0$. Hint: You don't have to use the equation for $P_{00}(t)$; why?

d) Check your answer in (b) with that in (c).

6.7) Consider an irreducible Markov process with n states and assume that the transition rate matrix $[Q] = [V][\Lambda][V]^{-1}$ where [V] is the matrix of right eigenvectors of [Q], $[\Lambda]$ is the diagonal matrix of eigenvalues of [Q], and the inverse of [Q] is the matrix of left eigenvectors.

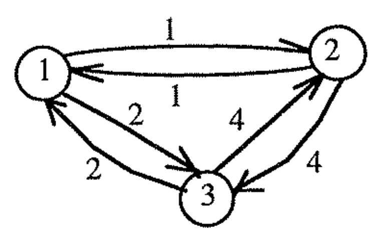
a) Consider the sampled time approximation to the process with an increment of size δ, and let $[W_\delta]$ be the transition matrix for the sampled time approximation. Express $[W_\delta]$ in terms of [V] and $[\Lambda]$.

b) Express $[W_\delta]^n$ in terms of [V] and $[\Lambda]$.

c) Expand $[W_\delta]^n$ in the same form as (25).

d) Let t be an integer multiple of δ, and compare $[W_\delta]^{t/\delta}$ to [P(t)]. Note: What you see from this is that λ_i in (25) is replaced by $(1/\delta)\ln(1+\delta\lambda_i)$. For the steady state term, $\lambda_1 = 0$, this causes no change, but for the other eigenvalues, there is a change that vanishes as $\delta \to 0$.

6.8) Consider the three state Markov process below; the number given on edge (i, j) is the transition rate from i to j, q_{ij}. Assume that the process is in steady state.



a) Is this process reversible?

b) Find p_i, the time average fraction of time spent in state i for each i.

c) Given that the process is in state i at time t, find the mean delay from t until the process leaves state i.

d) Find π_i, the time average fraction of all transitions that go into state i for each i.

e) Suppose the process is in steady state at time t. Find the steady state probability that the next state to be entered is state 1.

f) Given that the process is in state 1 at time t, find the mean delay until the process first returns to state 1.

g) Consider an arbitrary irreducible finite state Markov process in which $q_{ij} = q_{ji}$ for all i, j. Either show that such a process is reversible or find a counter example.

6.9) a) Consider an M/M/1 queueing system with arrival rate λ, service rate μ, $\mu > \lambda$. Assume that the queue is in steady state. Given that an arrival occurs at time t, find the probability that the system is in state i immediately *after* time t.

b) Assuming FCFS service, and conditional on i customers in the system immediately after the above arrival, characterize the time until the above customer departs as a sum of random variables.

c) Find the unconditional probability density of the time until the above customer departs. Hint: You know (from splitting a Poisson process) that the sum of a geometrically distributed number of IID exponentially distributed random variables is exponentially distributed. Use the same idea here.

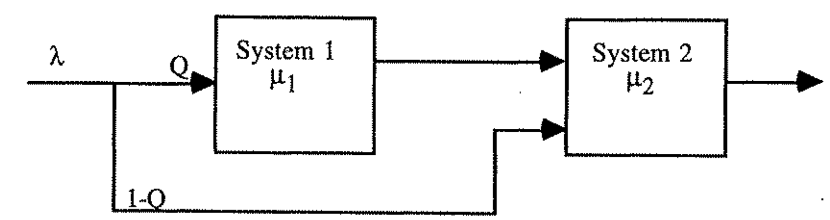
6.10) A small bookie shop has room for at most two customers. Potential customers arrive at a Poisson rate of 10 customers per hour; they enter if there is room and are turned away, never to return, otherwise. The bookie serves the admitted customers in order, requiring an exponentially distributed time of mean 4 minutes per customer.

a) Find the steady state distribution of number of customers in the shop.

b) Find the rate at which potential customers are turned away.

c) Suppose the bookie hires an assistant; the bookie and assistant, working together, now serve each customer in an exponentially distributed time of mean 2 minutes, but there is only room for one customer (i.e., the customer being served) in the shop. Find the new rate at which customers are turned away.

6.11) Consider the job sharing computer system illustrated below. Incoming jobs arrive from the left in a Poisson stream. Each job, independently of other jobs, requires pre-processing in system 1 with probability Q. Jobs in system 1 are served FCFS and the service times for successive jobs entering system 1 are IID with an exponential distribution of mean $1/\mu_1$. The jobs entering system 2 are also served FCFS and successive service times are IID with an exponential distribution of mean $1/\mu_2$. The service times in the two systems are independent of each other and of the arrival times. Assume that $\mu_1 > \lambda Q$ and that $\mu_2 > \lambda$. Assume that the combined system is in steady state.



a) Is the input to system 1 Poisson? Explain.

b) Are each of the two input processes coming into system 2 Poisson? Explain

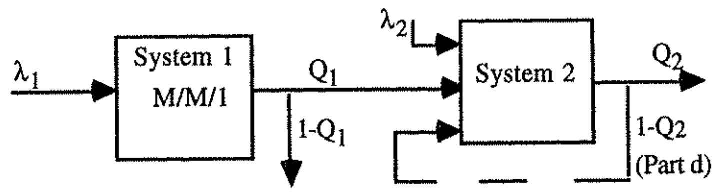
c) Considering the input process to system 1 and the two input processes to system 2, which are independent of each other? Explain carefully.

d) Give the joint steady state PMF of the number of jobs in the two systems. Explain briefly.

e) What is the probability that the first job to leave system 1 after time t is the same as the first job that entered the entire system after time t?

f) What is the probability that the first job to leave system 2 after time t both passed through system 1 and arrived at system 1 after time t?

6.12) Consider the following combined queueing system. The first queue system is M/M/1 with service rate μ_1. The second queue system has IID exponentially distributed service times with rate μ_2.



Each departure from system 1 independently goes to system 2 with probability Q_1 and leaves the system with probability 1-Q_1. System 2 has an additional Poisson input of rate λ_2, independent of inputs and outputs from the first system. Each departure from the second system independently leaves the combined system with probability Q_2 and re-enters system 2 with probability 1-Q_2. *For parts (a), (b), (c) assume that $Q_2 = 1$* (i.e., there is no feedback).

a) Characterize the process of departures from system 1 that enter system 2 and characterize the overall process of arrivals to system 2.

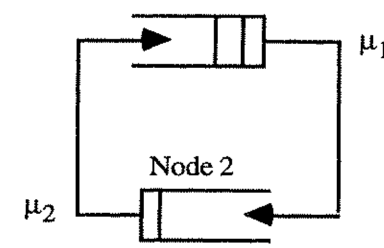
b) Assuming FCFS service in each system, find the steady state distribution of time that a customer spends in each system.

c) For a customer that goes through both systems, show why the time in each system is independent of that in the other; find the distribution of the combined system time for such a customer.

d) Now assume that $Q_2 < 1$. Is the departure process from the combined system Poisson? Which of the three input processes to system 2 are Poisson? Which of the input processes are independent? Explain your reasoning, but do not attempt formal proofs.

6.13) Suppose a Markov chain with transition probabilities $\{P_{ij}\}$ is reversible. For some given state, state 0 to be specific, suppose we change the transition probabilities out of state 0 from $\{P_{0j}\}$ to $\{P_{0j}'\}$. Assuming that $\{P_{ij}\}$ for all i, j with $i \neq 0$ are unchanged, what is the most general way in which we can choose $\{P_{0j}'\}$ so as to maintain reversibility? Your answer should be explicit about how the steady state probabilities $\{\pi_i\}$ are changed. Your answer should also indicate what this problem has to do with uniformization of reversible Markov processes, if anything. Hint: Given $\{P_{ij}\}$ a single additional parameter will suffice to specify $\{P_{0j}'\}$ for all j.

6.14) Consider the closed queueing network in the figure below. There are three customers who are doomed forever to cycle between node 1 and node 2. Both nodes use FCFS service and have exponentially distributed IID service times. The service times at one node are also independent of those at the other node and are independent of the customer being served. The server at node i has mean service time $1/\mu_i$, i = 1, 2. Assume to be specific that $\mu_2 < \mu_1$.



a) The system can be represented by a four state Markov process. Draw its graphical representation and label it with the individual states and the transition rates between them.

b) Find the steady state probability of each state.

c) Find the time average rate at which customers leave node 1.

d) Find the time average rate at which a given customer cycles through the system.

e) Is the Markov process reversible? Suppose that the backward Markov process is interpreted as a closed queueing network. What does a departure from node 1 in the forward process correspond to in the backward process? Can the transitions of a single customer in the forward process be associated with transitions of a single customer in the backward process?

6.15) Consider an M/G/1 queueing system with last come first serve (LCFS) service. That is, customers arrive according to a Poisson process of rate λ. A newly arriving customer interrupts the customer in service and enters service itself. When a customer is finished, it leaves the system and the customer that had been interrupted by the departing customer resumes service from where it had left off. For example, if customer 1 arrives at time 0 and requires 2 units of service, and customer 2 arrives at time 1 and requires 1 unit of service, then customer 1 is served from time 0 to 1; customer 2 is served from time 1 to 2 and leaves the system, and then customer 1 completes service from time 2 to 3. Let X_i be the service time required by the i^{th} customer; the X_i are IID random variables with expected value E[X]; they are independent of customer arrival times. Assume λ E[X] < 1.

a) Find the mean time between busy periods (i.e., the time until a new arrival occurs after the system becomes empty).

b) Find the time average fraction of time that the system is busy.

c) Find the mean duration, E[B], of a busy period. Hint: Use (a) and (b).

d) Explain briefly why the customer that starts a busy period remains in the system for the entire busy period; use this to find the expected system time of a customer given that that customer arrives when the system is empty.

e) Is there any statistical dependence between the system time of a given customer (i.e., the time from the customer's arrival until departure) and the number of customers in the system when the given customer arrives?

f) Show that a customer's expected system time is equal to E[B]. Hint: Look carefully at your answers to (d) and (e).

g) Let C be the expected system time of a customer conditional on the service time X of that customer being 1. Find (in terms of C) the expected system time of a customer conditional on X=2. Hint: Compare a customer with X=2 to two customers with X=1 each; repeat for arbitrary X=x.

h) Find the constant C. Hint: Use (f) and (g); don't do any tedious calculations.

6.16) Consider a queueing system with two classes of customers, type A and type B. Type A customers arrive according to a Poisson process of rate λ_A and customers of type B arrive according to an independent Poisson process of rate λ_B.

a) The queue has a single server with exponentially distributed IID service times of rate $\mu > \lambda_A + \lambda_B$. First come first serve service (FCFS) is used. Characterize the departure process of class A customers; explain carefully. Hint: Consider the combined arrival process and be judicious about how to select between A and B types of customers.

b) Suppose now that last come first serve (LCFS) service is used (i.e., whenever a new customer arrives, the server drops what it is doing and starts work on the new customer; when a customer departs, the server resumes service on the most recently arrived remaining customer). Characterize the departure process of class A customers; explain carefully.

c) Suppose now that LCFS service is used, but that now the single server requires independent exponentially distributed service times of rate μ_A for class A customers and rate μ_B for class B customers. Model this as a Markov process in which the state is the ordered set of customer classes in the queue and in service (see figure). What are the transition rates out of the state shown below? Is this process reversible? Assume $(\lambda_A/\mu_A) + (\lambda_B/\mu_B) < 1$.

Sample state of queueing system with type A customer most recently arrived, another type A next most recent, and customer B least recent.

d) Characterize the departure process of class A customers for the system of part (c); explain carefully.

6.17) Consider a k node Jackson type network with the modification that each node i has s servers rather than one server. Each server at i has an exponentially distributed service time with rate μ_i. The exogenous input rate to node i is $r_i = \lambda_0 Q_{0i}$ and each output from i is switched to j with probability Q_{ij} and switched out of the system with probability Q_{i0} (as in the notes). Let λ_i, $1 \le i \le k$, be the solution, for given λ_0, to

$$\lambda_j = \sum_{i=0}^{k} \lambda_i Q_{ij} \, ;$$

$1 \le j \le k$ and assume that $\lambda_i < s\mu_i$; $1 \le i \le k$. Show that the steady state probability of state **m** is

$$P(\mathbf{m}) = \prod_{i=1}^{k} P_i(m_i)$$

where $p_i(m_i)$ is the probability of state m_i in an (M,M,s) queue. Hint: Simply extend the argument in the text to the multiple server case.

6.18) Suppose a Markov process with the set of states A is reversible and has steady state probabilities p_i; $i \in A$. Let B be a subset of A and assume that the process is changed by setting $q_{ij} = 0$ for all $i \in B$, $j \notin B$. Assuming that the new process (starting in B and remaining in B) is irreducible, show that the new process is reversible and find its steady state probabilities.

6.19) Consider a sampled time M/D/m/m queueing system. The arrival process is Bernoulli with probability $\lambda \ll 1$ of arrival in each time unit. There are m servers; each arrival enters a server if a server is not busy and otherwise the arrival is discarded. If an arrival enters a server, it keeps the server busy for d units of time and then departs; d is some integer constant and is the same for each server.

Let n, $0 \le n \le m$ be the number of customers in service at a given time and let x_i be the number of time units that the i^{th} of those n customers (in order of arrival) has been in service. Thus the state of the system can be taken as $(n, \mathbf{x}) = (n, x_1, x_2, \ldots, x_n)$ where $0 \le n \le m$ and $1 \le x_1 < x_2 < \ldots < x_n \le d$.

Let $A(n, \mathbf{x})$ denote the next state if the present state is $(n, \mathbf{x})$ and a new arrival enters service. That is,

$$A(n, \mathbf{x}) = (n+1, 1, x_1+1, x_2+1, \ldots, x_n+1) \text{ for } n < m \text{ and } x_n < d \qquad (1e)$$
$$A(n, \mathbf{x}) = (n, 1, x_1+1, x_2+1, \ldots, x_{n-1}+1) \text{ for } n \le m \text{ and } x_n = d \qquad (2e)$$

That is, the new customer receives one unit of service by the next state time, and all the old customers receive one additional unit of service. If the oldest customer has received d units of service, then it leaves the system by the next state time. Note that it is possible for a customer with d units of service at the present time to leave the system and be replaced by an arrival at the present time (i.e., (2e) with n=m, x_n=d). Let $B(n, \mathbf{x})$ denote the next state if either no arrival occurs or if a new arrival is discarded.

$$B(n, \mathbf{x}) = (n, x_1+1, x_2+1, \ldots, x_n+1) \text{ for } x_n < d \qquad (3e)$$
$$B(n, \mathbf{x}) = (n-1, x_1+1, x_2+1, \ldots, x_{n-1}+1) \text{ for } x_n = d \qquad (4e)$$

a) Hypothesize that the *backward* chain for this system is also a sampled time M/D/m/m queueing system, but that the state $(n, x_1, \ldots, x_n)$ $(0 \le n \le m, 1 \le x_1 < x_2 < \ldots < x_n \le d)$ has a different interpretation: n is the number of customers as before, but x_i is now the remaining service required by customer i. Explain how this hypothesis leads to the following steady state equations: