We can evaluate P_{0j} by observing that the departure of that first arrival leaves j customers in this system iff j customers arrive during the service time of that first customer; i.e., the new state doesn't depend on how long the server waits for a new customer to serve, but only on the arrivals while that customer is being served. Letting g(u) be the density of the service time,

$$P_{0j} = \int_0^\infty \frac{g(u)(\lambda u)^j \exp(-\lambda u)}{j!}\, du \; ; \; j \geq 0 \qquad (72)$$

5.9 SUMMARY

This chapter extended the finite state Markov chain results of Chapter 4 to the case of countably infinite state spaces. It also provided an excellent example of how renewal processes can be used for understanding other kinds of processes. In section 5.1, the first passage time random variables were used to construct renewal processes with renewals on successive transitions to a given state. These renewal processes were used to rederive the basic properties of Markov chains using renewal theory as opposed to the algebraic Perron-Frobenius approach of Chapter 4. The central result of this was theorem 3, which showed that, for an irreducible chain, the states are positive recurrent iff the steady state equations, (14), have a solution. Also if (14) has a solution, it is positive and unique. We also showed that these steady state probabilities are, with probability 1, time averages for sample paths, and that, for an ergodic chain, they are limiting probabilities independent of the starting state.

We found that the major complications that result from countably infinite state spaces are, first, different kinds of transient behavior, and second, the possibility of null recurrent states. For finite state Markov chains, a state is transient only if it can reach some other state from which it can't return. For countably infinite chains, there is also the case, as in figure 5.1 for p>1/2, where the state just wanders away, never to return. Null recurrence is a limiting situation where the state wanders away and returns with probability 1, but with an infinite expected time. There is not much engineering significance to null recurrence; it is highly sensitive to modeling details over the entire infinite set of states. One usually uses countably infinite chains to simplify models; for example, if a buffer is very large and we don't expect it to overflow, we assume it is infinite. Finding out, then, that the chain is transient or null recurrent simply means that the modeling assumption was not very good.

Branching processes were introduced in section 5.2 as a model to study the growth of various kinds of elements that reproduce. In general, for these models (assuming p_0>0), there is one trapping state and all other states are transient. Figure 5.3 showed how to find the probability that the trapping state is entered by the n^{th} generation, and also the probability that it is entered eventually. If the expected number of offspring of an element is at most 1, then the population dies out with probability 1, and otherwise, the population dies out with some given probability q, and grows without bound with probability 1-q.

We next studied birth death Markov chains and reversibility. Birth death chains are widely used in queueing theory as sample time approximations for systems with Poisson arrivals and various generalizations of exponentially distributed service times. Eq. (29) gives their steady state probabilities if positive recurrent, and shows the condition under which they are positive recurrent. We showed that these chains are reversible if they are positive recurrent.

Theorems 5 and 6 provided a simple way to find the steady state distribution of reversible chains and also of chains where the backward chain behavior could be hypothesized or deduced. We used reversibility to show that M/M/1 and M/M/m Markov chains satisfy Burke's theorem for sampled time—namely that the departure process is Bernoulli, and that the state at any time is independent of departures before that time.

Round robin queueing was then used as a more complex example of how to use the backward process to deduce the steady state distribution of a rather complicated Markov chain; this also gave us added insight into the behavior of queueing systems and allowed us to show that, in the processor sharing limit, the distribution of number of customers is the same as that in an M/M/1 queue.

Finally, semi-Markov processes were introduced. Renewal theory again provided the key to analyzing these systems. Theorem 8 showed how to find the steady state probabilities of these processes, and it was shown that these probabilities could be interpreted both as time averages and, in the case of non-arithmetic transition times, as limiting probabilities in time.

For further reading on Markov chains with countably infinite state spaces, see [Fel66], [Ros83], or [Wol89]. Feller is particularly complete, but Ross and Wolff are somewhat more accessible. Harris, [Har63] is the standard reference on branching processes and Kelly, [Kel79] is the standard reference on reversibility. The material on round robin systems is from [Yat90] and is generalized there.

EXERCISES

5.1) Let $\{P_{ij}; i, j \geq 0\}$ be the set of transition probabilities for a countably infinite Markov chain. For each i, j, let $F_{ij}(n)$ be the probability that state j occurs sometime between time 1 and n inclusive, given X_0=i. For some given j, assume that $\{x_k; k\geq 0\}$ is a set of non-negative numbers satisfying $x_i = P_{ij} + \sum_{k\neq j} P_{ik}x_k$. Show that $x_i \geq F_{ij}(n)$ for all n and i, and hence that $x_i \geq F_{ij}(\infty)$ for all i. Hint: use induction.

5.2) a) For the Markov chain in figure 5.1, show that, for p≥1/2, $F_{00}(\infty) = 2(1-p)$ and show that $F_{i0}(\infty) = [(1-p)/p]^i$ for i≥1. Hint: First show that this solution satisfies (5) and then show that (5) has no smaller solution (see exercise 5.1). Note that you have shown that the chain is transient for p>1/2 and that it is recurrent for p=1/2.

b) Under the same conditions as part (a), show that $F_{ij}(\infty)$ equals 2(1-p) for j=i, equals $[(1-p)/p]^{i-j}$ for i>j, and equals 1 for i<j.

5.3) Let j be a transient state in a Markov chain and let j be accessible from i. Show that i is transient also. Interpret this as a form of Murphy's law (if something bad can happen, it will, where the bad thing is the lack of an eventual return). Note: Give a direct demonstration rather than using lemma 1.

5.4) Consider an irreducible positive recurrent Markov chain. Consider the renewal process $\{N_{jj}(t);\ t\geq 0\}$ where, given $X_0=j$, $N_{jj}(t)$ is the number of times that state j is visited from time 1 to t. For each $i\geq 0$, consider a renewal reward function $R_i(t)$ equal to 1 whenever the chain is in state i and equal to 0 otherwise. Let π_i be the time average reward.

a) Show that $\pi_i = 1/\bar{T}_{ii}$ for each i with probability 1.

b) Show that $\sum_i \pi_i = 1$. Hint: Consider $\sum_{i\leq M} \pi_i$ for any integer M.

c) Consider a renewal reward function $R_{ij}(t)$ that is 1 whenever the chain is in state i and the next state is state j. $R_{ij}(t) = 0$ otherwise. Show that the time average reward is equal to $\pi_i P_{ij}$ with probability 1. Show that $\pi_k = \sum_i \pi_i P_{ik}$ for all k.

5.5) Let $\{X_n;\ n\geq 0\}$ be a branching process with $X_0=1$. Let $\bar{Y}, \sigma^2$ be the mean and variance of the number of offspring of an individual.

a) Argue that $\lim_{n\to\infty} X_n$ exists with probability 1 and either has the value 0 (with probability $P_{10}(\infty)$) or the value ∞ (with probability $1 - P_{10}(\infty)$).

b) Show that $\mathrm{VAR}(X_n) = \sigma^2 \bar{Y}^{\,n-1}(\bar{Y}^{\,n} - 1)/(\bar{Y} - 1)$ for $\bar{Y} \neq 1$ and $\mathrm{VAR}(X_n) = n\sigma^2$ for $\bar{Y} = 1$.

5.6) There are n states and for each pair of states i and j, a positive number $d_{ij} = d_{ji}$ is given. A particle moves from state to state in the following manner: Given that the particle is in any state i, it will next move to any $j\neq i$ with probability P_{ij} given by

$$P_{ij} = \frac{d_{ij}}{\sum_{j\neq i} d_{ij}}$$

Assume that $P_{ii} = 0$ for all i. Show that the sequence of positions is a reversible Markov chain and find the limiting probabilities.

5.7) Consider a reversible Markov chain with transition probabilities P_{ij} and limiting probabilities π_i. Also consider the same chain truncated to the states 0, 1, ..., M. That is, the transition probabilities $\{P'_{ij}\}$ of the truncated chain are

$$P'_{ij} = \begin{cases} \dfrac{P_{ij}}{\sum_{k=0}^{M} P_{ik}} & ;\ 0 \leq i,j \leq M \\ 0\ ; & \text{elsewhere} \end{cases}$$

Show that the truncated chain is also reversible and has limiting probabilities given by

$$\bar{\pi}_i = \frac{\pi_i \sum_{j=0}^{M} P_{ij}}{\sum_{k=0}^{M} \pi_k \sum_{m=0}^{M} P_{km}}$$

5.8) A Markov chain (with states {0, 1, 2, ..., J-1} where J is either finite or infinite) has transition probabilities $\{P_{ij};\ i,j \geq 0\}$. Assume that $P_{0j} > 0$ for all $j > 0$ and $P_{j0} > 0$ for all $j > 0$. Also assume that for all i, j, k, $P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji}$.

a) Assuming also that all states are positive recurrent, show that the chain is reversible and find the steady state probabilities $\{\pi_i\}$ in simplest form.

b) Find a condition on $\{P_{0j};\ j\geq 0\}$ and $\{P_{j0};\ j\geq 0\}$ that is sufficient to ensure that all states are positive recurrent.

5.9) a) Use the birth and death model described in figure 5.4 to find the steady state probability mass function for the number of customers in the system (queue plus service facility) for the following queues:

i) M/M/1 with arrival probability $\lambda\delta$, service completion probability $\mu\delta$.

ii) M/M/m with arrival probability $\lambda\delta$, service completion probability $i\mu\delta$ for i servers busy, $1\leq i\leq m$.

iii) M/M/∞ with arrival probability $\lambda\delta$, service probability $i\mu\delta$ for i servers. Assume δ so small that $i\mu\delta < 1$ for all i of interest.

Assume the system is positive recurrent.

b) For each of the queues above give necessary conditions (if any) for the states in the chain to be (i) transient, (ii) null recurrent, (iii) positive recurrent.

c) For each of the queues find:

L = (steady state) mean number of customers in the system.
L = (steady state) mean number of customers in the queue.
W = (steady state) mean waiting time in the system.
W_q = (steady state) mean waiting time in the queue.

5.10) a) Given that an arrival occurs in the interval $(n\delta, (n+1)\delta)$ for the sampled time M/M/1 model in figure 5.5, find the conditional PMF of the state of the system at time $n\delta$ (assume n arbitrarily large and assume positive recurrence).

b) For the same model, find the expected number of customers seen in the system by the first arrival after time $n\delta$. Note: The purpose of this exercise is to make you cautious about the meaning of "the state seen by a random arrival."

5.11) Find the backward transition probabilities for the Markov chain model of age in figure 5.2. Draw the graph for the backward Markov chain, and interpret it as a model for residual life.

5.12) Consider the sample time approximation to the M/M/1 queue in figure 5.5.

a) Give the steady state probabilities for this chain (no explanations or calculations required—just the answer).

In parts (b) to (g) do not use reversibility and do not use Burke's theorem. Let X_n be the state of the system at time $n\delta$ and let D_n be a random variable taking on the value 1 if a departure occurs between $n\delta$ and $(n+1)\delta$, and the value 0 if no departure occurs. Assume that the system is in steady state at time $n\delta$.

b) Find $P(X_n=i, D_n=j)$ for $i \geq 0$, $j = 0,1$

c) Find $P(D_n=1)$

d) Find $P(X_n=i \mid D_n=1)$ for $i \geq 0$

e) Find $P(X_{n+1}=i \mid D_n=1)$ and show that X_{n+1} is statistically independent of D_n. Hint: Use part (d); also show that $P(X_{n+1}=i) = P(X_{n+1}=i \mid D_n=1)$ for all $i \geq 0$ is sufficient to show independence.

f) Find $P(X_{n+1}=i, D_{n+1}=j \mid D_n)$ and show that the pair of variables (X_{n+1}, D_{n+1}) is statistically independent of D_n.

g) For each $k>1$, find $P(X_{n+k}=i, D_{n+k}=j \mid D_{n+k-1}, D_{n+k-2}, ..., D_n)$ and show that the pair (X_{n+k}, D_{n+k}) is statistically independent of $(D_{n+k-1}, D_{n+k-2}, ..., D_n)$. Hint: Use induction on k; as a substep, find $P(X_{n+k}=i \mid D_{n+k-1}=1, D_{n+k-2}, ..., D_n)$ and show that X_{n+k} is independent of $D_{n+k-1}, D_{n+k-2}, ..., D_n$.

h) What do your results mean relative to Burke's theorem?

5.13) Let $\{X_n, n \geq 1\}$ denote an irreducible Markov chain having a countable state space. Now consider a new stochastic process $\{Y_n, n \geq 0\}$ that only accepts values of the Markov chain that are between 0 and some integer m. For instance, if m = 3 and $X_1 = 1$, $X_2 = 3$, $X_3 = 5$, $X_4 = 6$, $X_5 = 2$, then $Y_1 = 1$, $Y_2 = 3$, $Y_3 = 2$.

a) Is $\{Y_n, n \geq 0\}$ a Markov chain? Explain briefly.

b) Let π_j denote the proportion of time that $\{X_n, n \geq 1\}$ is in state j. If $\pi_j > 0$ for all j, what proportion of time is $\{Y_n, n \geq 0\}$ in each of the states 0, 1, ..., m?

c) Suppose $\{X_n\}$ is null recurrent and let $\pi_i(m)$, i = 0,1, ..., m denote the long-run proportions for $\{Y_n, n \geq 0\}$. Show that

$$\pi_j(m) = \pi_i(m)E[\text{time the X process spends in j between returns to i}], \quad j \neq i.$$

5.14) Verify that (48) is satisfied by the hypothesized solution to π in (52). Also show that the equations involving the idle state ϕ are satisfied.

5.15) Replace the state $\mathbf{m} = (m, z_1, ..., z_m)$ in section 5.6 with an expanded state $\mathbf{m} = (m, z_1, w_1, z_2, w_2, ..., z_m, w_m)$ where m and $\{z_i; 1 \leq i \leq m\}$ are as before and $w_1, w_2, ..., w_m$ are the original service requirements of the m customers.

a) Hypothesizing the same backward round robin system as hypothesized in section 5.6, find the backward transition probabilities and give the corresponding equations to (46–49) for the expanded state description.

b) Solve the resulting equations to show that

$$\pi_{\mathbf{m}} = \pi_\phi\left(\frac{\lambda\delta}{1-\lambda\delta}\right)^m \prod_{j=1}^{m} f(w_j)$$

c) Show that the probability that there are m customers in the system, and that those customers have original service requirements given by $w_1, ..., w_m$, is

$$P(m, w_1,...,w_m) = \pi_\phi\left(\frac{\lambda\delta}{1-\lambda\delta}\right)^m \prod_{j=1}^{m} (w_j-1)f(w_j)$$

d) Given that a customer has original service requirement w, find the expected time that customer spends in the system.

5.16) A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability 1/3 and to 3 with probability 2/3. From 3 it always goes to 1. The mean time between locations i and j are $t_{12} = 20$, $t_{13} = 30$, $t_{23} = 30$ $(t_{ij} = t_{ji})$.

i) What is the (limiting) probability that the taxi's most recent stop was at location i, i = 1, 2, 3?

ii) What is the (limiting) probability that the taxi is heading for location 2?

iii) What fraction of time is the taxi traveling from 2 to 3. Note: Upon arrival at a location the taxi immediately departs.

5.17) Consider an M/G/1 queueing system with Poisson arrivals of rate λ and expected service time E[X]. Let $\rho = \lambda E[X]$ and assume $\rho < 1$. Consider a semi-Markov process model of the M/G/1 queueing system in which transitions occur on departures from the queueing system and the state is the number of customers immediately following a departure.

a) Suppose a colleague has calculated the steady state probabilities $\{p_i\}$ of being in state i for each $i \geq 0$. For each $i \geq 0$, find the steady state probability π_i of state i in the embedded Markov chain. Give your solution as a function of ρ, p_i, and p_0.

b) Calculate p_0 as a function of ρ.

c) Find π_i as a function of ρ and p_i.

d) Is p_i the same as the steady state probability that the queueing system contains i customers at a given time? Explain carefully.

5.18) Consider an M/G/1 queue in which the arrival rate is λ and the service time distribution is uniform (0,2W) with $\lambda W < 1$. Define a semi-Markov chain following the framework for the M/G/1 queue in section 5.8.

a) Find P_{0j}; $j \geq 0$.

b) Find P_{ij} for $i > 0$; $j \geq i-1$.

5.19) Consider a semi-Markov process for which the embedded Markov chain is irreducible and positive recurrent. Assume that the distribution of inter-renewal intervals for one state j is arithmetic with span d. Show that the distribution of inter-renewal intervals for all states is arithmetic with the same span.

Chapter 6

Markov Processes with Countable State Spaces

6.1 INTRODUCTION

A *Markov process* with a countable state space is a special case of a semi-Markov process in which, first, the interval between successive transitions has an exponential distribution, and second, that interval is independent of the next state. Thus, we can take the set of possible states as {0, 1, 2, ...} and the process as $\{X(t), t \geq 0\}$, where for each real $t \geq 0$, X(t) is the state of the process at time t. The random variables $S_1, S_2, \ldots$ denote the successive epochs at which the process makes state transitions, and X_n denotes the state entered at time S_n, i.e., $X_n = X(S_n)$ and $X(t) = X_n$ for $S_n \leq t < S_{n+1}$. Let $S_0=0$, and let $X_0 = X(0) = X(S_0)$ denote the initial state. The *embedded Markov chain* $\{X_n, n \geq 0\}$ has transition probabilities $\{P_{ij}, i \geq 0, j \geq 0\}$, and we assume that $P_{ii}=0$ for all i (i.e., there are no self transitions). The assumption $P_{ii}=0$ will be removed later when we talk about uniformized Markov processes. The intervals $U_n = S_n - S_{n-1}$ between successive transition epochs satisfy

$$P(U_n \leq x \mid X_{n-1}=i, X_n=j\} = 1 - \exp(-\nu_i x) \tag{1}$$

where, for each i, ν_i is a positive number called the *transition rate* out of state i. Conditional on X_{n-1}, the interval U_n is independent of X_n and also independent of all earlier inter-transition intervals and states.

Let Y(t) denote the residual time from t until the next transition after t. Given that X(t)=i, the memoryless property of the exponential distribution implies that Y(t) has an exponential distribution, $1 - \exp(-\nu_i t)$, and that Y(t) is independent of the next state and independent of $X(\tau)$ for all $\tau < t$. Thus, for all $j \neq i$,

$$P[Y(t) \leq x, X(t+Y(t))=j \mid X(t)=i, \{X(\tau); \tau < t\}] = P_{ij}[1-\exp(-\nu_i x)] \tag{2}$$

For x sufficiently small, the probability of two transitions in (0,x] is negligible, so the probability on the left is that of a transition to state j in (0,x]. Using δ in place of x, this becomes