problem solving techniques. The strong law of large numbers requires mathematical maturity, and might be postponed to Chapter 3 when it is first used.

There are too many texts on elementary probability to mention here, and most of them serve to give added understanding and background to the material here. [Ros94] and [Dra67] are both quite readable. [Kol50] is of historical interest (and is also readable) as the translation of the 1933 book that first put probability on a firm mathematical basis. [Fel68] is an extended and elegant treatment of elementary material from a mature point of view.

EXERCISES

1.1) The text shows that, for a non-negative random variable X with distribution function $F_X(x)$, $E[X] = \int_0^\infty [1-F_X(x)]dx$.

a) Write this integral as a sum for the special case in which X is a non-negative integer random variable.

b) Generalize the above integral for the case of an arbitrary (rather than non-negative) random variable Y with distribution function $F_Y(y)$; use a graphical argument.

c) Find $E[|Y|]$ by the same type of argument.

d) For what value of α is $E[|Y-\alpha|]$ minimized? Use a graphical argument again.

1.2) Let X be a random variable with distribution function $F_X(x)$. Find the distribution function of the following random variables.

a) The maximum of n IID random variables with distribution function $F_X(x)$.

b) The minimum of n IID random variables with distribution $F_X(x)$.

c) The difference of the random variables defined in (a) and (b); assume X has a density $f_X(x)$.

1.3) a) Let $X_1, X_2, ..., X_n$ be random variables with expected values $\overline{X}_1, ..., \overline{X}_n$. Prove that $E[X_1+...+X_n] = \overline{X}_1+...+\overline{X}_n$. Do not assume that the random variables are independent.

b) Now assume that $X_1, ..., X_n$ are statistically independent and show that the expected value of the product is equal to the product of the expected values.

c) Again assuming that $X_1, ..., X_n$ are statistically independent, show that the variance of the sum is equal to the sum of the variances.

1.4) Suppose $X_1, X_2, X_3, ...$ is a sequence of continuous IID random variables. X_n, for a given n>1, is called a local minimum of the sequence if $X_n \le X_{n+1}$, $X_n \le X_{n-1}$. Find the probability that X_n is a local minimum. Hint: No computation is necessary—use symmetry.

1.5) Let $X_1, X_2, ..., X_n, ...$ be a sequence of independent identically distributed (IID) continuous random variables with the common probability density function $f_X(x)$; note that $P(X = \alpha) = 0$ for all α and that $P(X_1 = X_2) = 0$.

a) Find $P(X_1 \le X_2)$ (give a numerical answer, not an expression; no computation is required and a one or two line explanation should be adequate).

b) Find $P(X_1 \le X_2; X_1 \le X_3)$ (in other words, find the probability that X_1 is the smallest of X_1, X_2, X_3; again, think—don't compute).

c) Let the random variable N be the index of the first r.v. in the sequence to be less than X_1; that is, $P(N=n) = P(X_1 \le X_2; X_1 \le X_3; ...X_1 \le X_{n-1}; X_1 > X_n)$. Find $P(N > n)$ as a function of n. Hint: Generalize part (b).

d) Show that $E[N] = \infty$.

1.6) a) Assume that X is a discrete random variable taking on values $a_1, a_2, ...$, and let $Y = g(X)$. Let $b_i = g(a_i)$, $i \ge 1$ be the ith value taken on by Y. Show that $E[Y] = \Sigma_i b_i P_Y(b_i) = \Sigma_i g(a_i) P_X(a_i)$.

b) Let X be a continuous random variable with density $f_X(x)$ and let g be differentiable and monotonic increasing. Show that $E[Y] = \int y f_Y(y)\, dy = \int g(x) f_X(x)\, dx$.

1.7) a) Show that, for uncorrelated random variables, the expected value of the product is equal to the product of the expected values (X and Y are uncorrelated if

$$E[(X-E[X])(Y-E[Y])] = 0).$$

b) Show that if X and Y are uncorrelated, then the variance of X + Y is equal to the variance of X plus the variance of Y.

c) Show that if $X_1, ..., X_n$ are uncorrelated, the the variance of the sum is equal to the sum of the variances.

d) Show that independent random variables are uncorrelated.

e) Let X,Y be identically distributed ternary valued random variables with the probability assignment $P(X = 1) = P(X = -1) = 1/4$; $P(X = 0) = 1/2$. Find a *simple* joint probability assignment such that X and Y are uncorrelated but dependent.

f) You have seen that the moment generating function of a sum of independent random variables is equal to the product of the individual moment generating functions. Give an example where this is false if the variables are uncorrelated but dependent.

1.8) Suppose X has the Poisson PMF, $P(X = n) = \lambda^n \exp(-\lambda)/n!$ for $n \ge 0$ and Y has the Poisson PMF, $P(Y = n) = \mu^n \exp(-\mu)/n!$ for $n \ge 0$. Find the distribution of $Z = X + Y$ and find the conditional distribution of Y conditional on Z = n.

1.9) a) Suppose X, Y and Z are binary random variables, each taking on the value 0 with probability 1/2 and the value 1 with probability 1/2. Find an example in which X, Y, Z are statistically dependent but are *pairwise* statistically independent (i.e., X, Y are statistically independent, X, Z are statistically independent, and Y, Z are statistically independent). Give $P_{XYZ}(x,y,z)$ for your example.

b) Is pairwise statistical independence enough to ensure that

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

for a set of random variables $X_1, ..., X_n$?

1.10) Show that E[X] is the value of z that minimizes $E[(X-z)^2]$.

1.11) A computer system has n users, each with a unique name and password. Due to a software error, the n passwords are randomly permuted internally (i.e., each of the n! possible permutations are equally likely). Only those users lucky enough to have had their passwords unchanged in the permutation are able to continue using the system.

a) What is the probability that a particular user, say user 1, is able to continue using the system?

b) What is the expected number of users able to continue using the system? Hint: Let X_i be a random variable with the value 1 if user i can use the system and 0 otherwise.

1.12) Suppose the random variable X is continuous and has the distribution function $F_X(x)$. Consider another random variable $Y = F_X(X)$. That is, for any sample point α such that $X(\alpha) = x$, we have $Y(\alpha) = F_X(x)$. Show that Y is uniformly distributed in the interval 0 to 1.

1.13) Let Z be an integer valued random variable with the PMF $P_Z(n) = 1/k$ for $0 \le n \le k-1$. Find the mean, variance, and moment generating function of Z. Hint: The elegant way to do this is to let U be a uniformly distributed continuous random variable over (0,1] that is independent of Z. Then U+Z is uniform over (0,k]. Use the known results about U and U+Z to find the mean, variance, and mgf for Z.

1.14) Let $\{X_n\ ;\ n \ge 1\}$ be a sequence of independent but not identically distributed random variables. We say that the weak law of large numbers holds for this sequence if for all $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{E[S_n]}{n}\right| \ge \varepsilon\right) = 0 \quad \text{where } S_n = X_1 + X_2 + ... + X_n \qquad \text{(a)}$$

a) Show that (a) holds if there is some constant A such that $VAR(X_n) \le A$ for all n.

b) Suppose that $VAR(X_n) \le A\, n^{1-\alpha}$ for some $\alpha < 1$ and for all n. Show that (a) holds in this case.

1.15) Let $\{X_i;\ i \ge 1\}$ be IID Bernoulli random variables. Let $P(X_i=1) = \delta$, $P(X_i=0) = 1-\delta$. Let $S_n = X_1 + ... + X_n$. Let m be an arbitrary but fixed positive integer. Think! then evaluate the following and explain your answers:

a) $\lim_{n\to\infty} \sum_{i:\ n\delta - m \le i \le n\delta + m} P(S_n = i)$

b) $\lim_{n\to\infty} \sum_{i:\ 0 \le i \le n\delta + m} P(S_n = i)$

c) $\lim_{n\to\infty} \sum_{i:\ n(\delta - 1/m) \le i \le n(\delta + 1/m)} P(S_n = i)$

1.16) Let $\{X_i;\ i \ge 1\}$ be IID random variables with mean 0 and infinite variance. Assume that $E[|X_i|^{1+h}] = \beta$ for some given h, $0<h<1$ and some given β. Let $S_n = X_1 + ... + X_n$.

a) Show that $P(|X_i| \ge y) \le \beta\, y^{-1-h}$

b) Let $\{\tilde{X}_i;\ i \ge 1\}$ be truncated variables $\tilde{X}_i = \begin{cases} b; & X_i \ge b \\ X_i; & -b \le X_i \le b \\ -b; & X_i \le -b \end{cases}$

Show that $E[\tilde{X}^2] \le \dfrac{2\,\beta\, b^{1-h}}{1-h}$

Hint: For a non-negative r.v. Z, $E[Z^2] = \int_0^\infty 2\, z\, P(Z \ge z)\, dz$ (you can establish this, if you wish, by integration by parts).

c) Let $\tilde{S}_n = \tilde{X}_1 + ... + \tilde{X}_n$. Show that $P(S_n \ne \tilde{S}_n) \le n\,\beta\, b^{-1-h}$

d) Show that $P\left(\left|\frac{S_n}{n}\right| \ge \varepsilon\right) \le \beta\left[\frac{2b^{1-h}}{(1-h)n\,\varepsilon^2} + \frac{n}{b^{1+h}}\right]$

e) Optimize your bound with respect to b. How fast does this optimized bound approach 0 with increasing n?

1.17) A town starts a mosquito control program and we let the random variable Z_n be the number of mosquitos at the end of the n^{th} year (n = 0, 1, 2, ...). Let X_n be the growth rate of mosquitos in year n; i.e., $Z_n = X_n Z_{n-1}$; $n \ge 1$. Assume that $\{X_n;\ n \ge 1\}$ is a sequence of IID random variables with the PMF $P(X = 2) = 1/2$; $P(X = 1/2) = 1/4$; $P(X=1/4) = 1/4$. Suppose that Z_0, the initial number of mosquitos, is some known constant and assume for simplicity and consistency that Z_n can take on non-integer values.

a) Find $E[Z_n]$ as a function of n and find $\lim_{n\to\infty} E[Z_n]$.

b) Let $W_n = \log_2 X_n$. Find $E[W_n]$ and $E[\log_2 (Z_n/Z_0)]$ as a function of n.

c) There is a constant α such that $\lim_{n\to\infty} (1/n)[\log_2(Z_n/Z_0)] = \alpha$ with probability 1. Find α and explain how this follows from the strong law of large numbers

d) Using (c), show that $\lim_{n\to\infty} Z_n = \beta$ with probability 1 for some constant β and evaluate β.

e) Explain carefully how the result in (a) and the result in (d) are possible. What you should learn from this problem is that the expected value of the log of a product of IID random variables is more significant that the expected value of the product itself.

1.18) Use figure 1.4 to verify Eq. (20). Hint: Show that $y\ P(Y \geq y) \leq \int_{z \geq y} z dF_Y(z)$ and show that $\lim_{y \to \infty} \int_{z \geq y} z dF_Y(z) = 0$ if E[Y] is finite.

1.19) Show that $\Pi_{m \geq n} (1-1/m) = 0$. Hint: Note that $(1-1/m) = \exp(\ln(1-1/m) \leq \exp(-1/m)$.

NOTES

1. One must add the axioms of set theory to this and specify the class of events.
2. A set is countable if it has a finite number of elements or if its elements can be put into one to one correspondence with the positive integers. See any elementary text on set theory.
3. The *sup*, or *supremum*, of a set of numbers is the smallest number greater than or equal to all members of the set. It is essentially the maximum of the set, but takes care of situations where the max doesn't exist. For example, the sup of real numbers x satisfying $x<2$ is 2, whereas there is no maximum x less than 2. The *inf*, or *infimum*, is defined similarly as the largest number less than or equal to all members of the set. It is essentially the minimum of the set.
4. Feller, *An Introduction to Probability Theory and its Application*, vol. I and II, Wiley, 1968 and 1966.
5. Central limit theorems also hold in many of these more general situations, but they usually do not have quite the generality of the laws of large numbers.
6. Proofs and sections marked with an asterisk, while instructive, can be omitted without loss of continuity.

Chapter 2

Poisson Processes

2.1 INTRODUCTION

A Poisson process is a simple and widely used stochastic process for modeling the times at which arrivals enter a system. We usually look at arrivals after some starting time, say t=0. Figure 2.1 illustrates some of the different ways to characterize random arrivals over the positive time axis. The sequence of times at which arrivals occur is denoted by the random variables $\{S_1, S_2, ...\}$. We usually refer to a point on the time axis at which something happens as an *epoch*, and thus we refer to S_n as the epoch of the n^{th} arrival, or the n^{th} arrival epoch. In principle, an arrival process can be characterized by a rule specifying the joint distribution functions of $\{S_1, S_2, ..., S_n\}$ for all $n \geq 1$, but usually these distribution functions are derived in terms of other random variables.

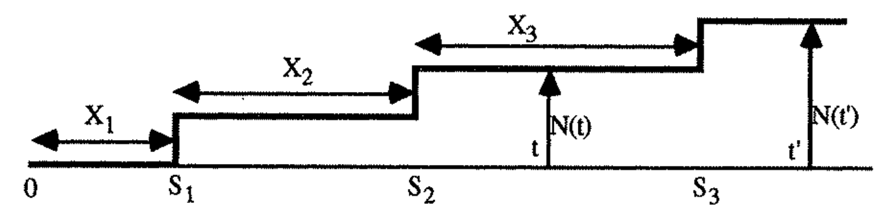


Figure 2.1. An arrival process and its arrival epochs $(S_1, S_2, ...)$, its inter-arrival intervals $(X_1, X_2, ...)$, and its counting process $(\{N(t); t \geq 0\})$.

An arrival process over the positive time axis can also be described by the inter-arrival intervals, denoted $\{X_1, X_2, ...\}$. For $n \geq 2$, X_n is the interval between the n-1[st] and the n^{th} arrival epoch, i.e., $X_n = S_n - S_{n-1}$. By convention, $X_1 = S_1$. It follows that the n^{th} arrival epoch can be expressed in terms of the inter-arrival intervals as

$$S_n = \sum_{i=1}^{n} X_i \qquad (1)$$

A rule specifying the joint distribution function of $\{X_1, ..., X_n\}$ for all $n \geq 1$ specifies the arrival process. Renewal processes, the topic of Chapter 3, are usually specified di-

$$P(S_{i+1}>s_{i+1} \mid N(t)=n,\ S_i=s_i) = \left[\frac{t-s_{i+1}}{t-s_i}\right]^{n-i} \qquad (30)$$

We note that this is independent of $S_1, ..., S_{i-1}$. As a check, one can find the conditional densities from (30) and multiply them all together to get back to (24) (see Exercise 2.21). One can also find the distribution of each S_i conditioned on N(t)=n but unconditioned on $S_1, S_2, ..., S_{i-1}$. The density for this is calculated by looking at n uniformly distributed random variables in (0,t]. The probability that one of these lies in the interval (x, x+dt] is (n dt)/t. Out of the remaining n-1, the probability that i-1 lie in the interval (0,x] is given by the binomial distribution with probability of success x/t. Thus the desired density is

$$f_{S_i}(x \mid N(t)=n)\, dt = \frac{x^{i-1}(t-x)^{n-i}(n-1)!}{t^{n-1}(n-i)!(i-1)!}\,\frac{n\, dt}{t}$$

$$f_{S_i}(x \mid N(t)=n) = \frac{x^{i-1}(t-x)^{n-i}n!}{t^{n}(n-i)!(i-1)!} \qquad (31)$$

2.6 SUMMARY

We started the chapter with three equivalent definitions of a Poisson process—first as a renewal process with exponentially distributed inter-renewal intervals, second as a stationary and independent increment counting process with Poisson distributed arrivals in each interval, and third essentially as a limit of a Bernoulli process. We saw that each definition provided its own insights into the properties of the process. We emphasized the importance of the memoryless property of the exponential distribution, both as a useful tool in problem solving and as an underlying reason why the Poisson process is so simple.

We next showed that the sum of independent Poisson processes is again a Poisson process. We also showed that if the arrivals in a Poisson process were independently routed to different locations with some fixed probability assignment, then the arrivals at each of these locations formed independent Poisson processes. This ability to view independent Poisson processes either independently or as a splitting of a combined process is a powerful technique to find almost trivial solutions to many problems.

It was next shown that a non-homogeneous Poisson process could be viewed as a (homogeneous) Poisson process on a non-linear time scale. This allows all the properties of (homogeneous) Poisson properties to be applied directly to the non-homogeneous case. The simplest and most useful result from this is (20), showing that the number of arrivals in any interval has a Poisson PMF. This result was used to show that the number of customers in service at any given time τ in an M/G/∞ queue has a Poisson PMF with a mean approaching λ times the expected service time as $\tau\to\infty$.

Finally we looked at the distribution of arrivals conditional on n arrivals in the interval (0,t]. It was found that these arrivals had the same joint distribution as the order statistics of n uniform IID random variables in (0,t]. By using symmetry and going

back and forth between the uniform variables and the Poisson process arrivals, we found the distribution of the interarrival times, of the arrival epochs, and of various conditional distributions.

EXERCISES

2.1) a) Find the Erlang densities $f_{S_n}(t)$ by convolving $f_X(x) = \lambda\exp(-\lambda x)$ with itself n times.

b) Find the moment generating function of X (or find the Laplace transform of $f_X(x)$), and use this to find the moment generating function (or Laplace transform) of $S_n = X_1+X_2+...+X_n$. Invert your result to find $f_{S_n}(t)$.

c) Find the mean, variance, and moment generating function of N(t), as given by (9). Show that the sum of two independent Poisson random variables is again Poisson.

2.2) The purpose of this exercise is to give an alternate derivation of the Poisson distribution for N(t), the number of arrivals in a Poisson process up to time t; let λ be the rate of the process.

a) Find the conditional probability $P(N(t) = n \mid S_n = \tau)$ for all $\tau \le t$.

b) Using the Erlang density for S_n, use (a) to find $P(N(t) = n)$.

2.3) Assume that a counting process $\{N(t);\ t \ge 0\}$ has the independent and stationary increment properties and satisfies (9) (for all t > 0).

a) Let X_1 be the epoch of the first arrival and X_n be the interarrival time between the n-1st and the nth arrival. Show that $P(X_1>x) = e^{-\lambda x}$.

b) Let S_{n-1} be the epoch of the n-1st arrival. Show that $P(X_n>x \mid S_{n-1}=\tau) = e^{-\lambda x}$.

c) Show that, for each n>1, $P(X_n>x) = e^{-\lambda x}$ and X_n is independent of S_{n-1}.

d) Argue that X_n is independent of $X_1, X_2, ..., X_{n-1}$.

2.4) Assume that a counting process $\{N(t);\ t \ge 0\}$ has the independent and stationary increment properties and satisfies (for all $t > 0$, $\delta > 0$)

$$P(\tilde{N}(t,t+\delta) = 0) = 1 - \lambda\delta + o(\delta)$$

$$P(\tilde{N}(t,t+\delta) = 1) = \lambda\delta + o(\delta)$$

$$P(\tilde{N}(t,t+\delta) > 1) = o(\delta)$$

a) Let $F_0(\tau) = P(N(\tau) = 0)$ and show that $F'_0(\tau) = -\lambda F_0(\tau)$.

b) Show that X_1, the time of the first arrival, is exponential with parameter λ.

c) Let $F_n(\tau) = P(\tilde{N}(t, t+\tau) = 0 \mid S_{n-1} = t)$ and show that $F'_n(\tau) = -\lambda F_n(\tau)$.

d) Argue that X_n is exponential with parameter λ and independent of earlier arrival times.

2.5) Let t>0 be an arbitrary time, let Z_1 be the duration of the interval from t until the next arrival after t, and let Z_m, for each m>1, be the interarrival time from the epoch of the m-1th arrival after t until the mth arrival.

a) Given that $N(t) = n$, explain why $Z_m = X_{m+n}$ for $m>1$ and $Z_1 = X_{n+1}-t+S_n$.
b) Conditional on $N(t)=n$ and $S_n=\tau$, show that $Z_1, Z_2, \ldots$ are IID.
c) Show that $Z_1, Z_2, \ldots$ are IID.

2.6) Consider a "baby Bernoulli" approximation to a Poisson process. X_i is the number of arrivals in the interval $(i\delta-\delta, i\delta]$, and we assume $\{X_i; i\geq 1\}$ is IID with $P(X_i=1) = \delta\lambda$, and $P(X_i=0) = 1-\delta\lambda$. Let $N(k\delta) = X_1+X_2+\ldots+X_k$ be the number of arrivals in $(0, k\delta]$ according to the baby Bernoulli approximation.

a) Show that $P(N(k\delta)=n) = \binom{k}{n}(\lambda\delta)^n(1-\lambda\delta)^{k-n}$

b) Let $t = k\delta$, and consider holding t fixed as $\delta\to 0$ and $k\to\infty$. Show that for any given n,

$$P(N(t)=n) = (1+\nu(\delta))\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

where $\nu(\delta)$ is a function of δ satisfying $\lim_{\delta\to 0} \nu(\delta) = 0$.

Hint: Show that $\frac{k!}{(k-n)!} = k^n \exp\left[\sum_{i=1}^{n-1} \ln(1 - \frac{i}{k})\right] = (1+\nu(\delta))\, k^n$

Show that $(1-\lambda\delta)^{k-n} = \exp\left[(k-n)\ln(1-\lambda\delta)\right] = (1+\nu(\delta)) \exp(-\lambda t)$.

2.7) Let $\{N(t); t\geq 0\}$ be a Poisson process of rate λ.
a) Find the joint probability mass function (PMF) of $N(t)$, $N(t+s)$ for $s>0$.
b) Find $E[N(t)\cdot N(t+s)]$ for $s>0$.
c) Find $E[\tilde{N}(t_1,t_3)\cdot\tilde{N}(t_2,t_4)]$ where $\tilde{N}(t,\tau)$ is the number of arrivals in $(t,\tau]$ and $t_1 < t_2 < t_3 < t_4$.

2.8) An experiment is independently performed N times where N is a Poisson random variable of mean λ. Let $\{a_1, a_2, \ldots, a_K\}$ be the set of elementary outcomes of the experiment and let P_k, $1\leq k\leq K$, denote the probability of a_k.
a) Let N_i denote the number of experiments performed for which the output is a_i. Find the PMF for N_i $(1 \leq i \leq K)$. Hint: no calculation is necessary.
b) Find the PMF for $N_1 + N_2$.
c) Find the conditional PMF for N_1 given that $N = n$.
d) Find the conditional PMF for $N_1 + N_2$ given that $N = n$.
e) Find the conditional PMF for N given that $N_1 = n_1$.

2.9) Starting from time 0, northbound buses arrive at 77 Mass. Avenue according to a Poisson process of rate λ. Passengers arrive according to an independent Poisson process of rate μ. When a bus arrives, all waiting customers instantly enter the bus and subsequent customers wait for the next bus.
a) Find the PMF for the number of customers entering a bus (more specifically, for any given m, find the PMF for the number of customers entering the m^{th} bus).

b) Find the PMF for the number of customers entering the m^{th} bus given that the interarrival interval between bus m-1 and bus m is x.
c) Given that a bus arrives at time 10:30 PM, find the PMF for the number of customers entering the next bus.
d) Given that a bus arrives at 10:30 PM and no bus arrives between 10:30 and 11, find the PMF for the number of customers on the next bus.
e) Find the PMF for the number of customers waiting at some given time, say 2:30 PM (assume that the processes started infinitely far in the past). Hint: Think of what happens moving backward in time from 2:30 PM.
f) Find the PMF for the number of customers getting on the next bus to arrive after 2:30. Hint: this is different from part (a); look carefully at part (e).
g) Given that I arrive to wait for a bus at 2:30 PM, find the PMF for the number of customers getting on the next bus.

2.10) Eq. (31) in chapter 2 gives $f_{S_i}(x \mid N(t) = n)$, the density of random variable S_i conditional on $N(t) = n$ for $n \geq i$. Multiply this expression by $P(N(t) = n)$ and sum over n to find $f_{S_i}(x)$; verify that your answer is indeed the Erlang density.

2.11) Consider generalizing the bulk arrival process in figure 2.4. Assume that the epochs at which arrivals occur form a Poisson process $\{N(t); t\geq 0\}$ of rate λ. At each arrival epoch, S_n, the number of arrivals, Z_n, satisfies $P(Z_n=1) = p$, $P(Z_n=2) = 1-p$. The variables Z_n are IID.
a) Let $\{N_1(t); t\geq 0\}$ be the counting process of the epochs at which single arrivals occur. Find the PMF of $N_1(t)$ as a function of t. Similarly, let $\{N_2(t); t\geq 0\}$ be the counting process of the epochs at which double arrivals occur. Find the PMF of $N_2(t)$ as a function of t.
b) Let $\{N_B(t); t\geq 0\}$ be the counting process of the total number of arrivals. Give an expression for the PMF of $N_B(t)$ as a function of t.

2.12) a) For a Poisson counting process of rate λ, find the joint probability density of $S_1, S_2, \ldots, S_{n-1}$ conditional on $S_n = t$. Use the same technique for the condition $S_n=t$ as in (24) for the condition $N(t) = n$.
b) Find $P(X_1 > \tau \mid S_n = t)$.
c) Find $P(X_i > \tau \mid S_n = t)$ for $1 \leq i \leq n$.
d) Find the density $F_{S_i}(x|S_n = t)$ for $1 < i < n-1$.
e) Give an explanation for the striking similarity between the condition $N(t) = n-1$ and the condition $S_n = t$.

2.13) a) For a Poisson process of rate λ, find $P(N(t) = n \mid S_1 = \tau)$ for $t > \tau$, $n > 1$.
b) Using this, find $f_{S_1}(\tau \mid N(t) = n)$
c) Check your answer against (25).

2.14) Consider a counting process in which the rate is a random variable Λ with probability density $f_\Lambda(\lambda) = \alpha\, e^{-\alpha\lambda}$ for $\lambda > 0$. Conditional on a given sample value λ for the rate, the counting process is a Poisson process of rate λ (i.e., nature first chooses a sample value λ and then generates a sample function of a Poisson process of that rate λ).

a) What is $P(N(t)=n \mid \Lambda=\lambda)$, where $N(t)$ is the number of arrivals in the interval $(0, t]$ for some given $t > 0$?

b) Show that $P(N(t)=n)$, the unconditional PMF for $N(t)$, is given by

$$P\big(N(t)=n\big)=\frac{\alpha t^n}{(t+\alpha)^{n+1}}$$

c) Find $f_\Lambda(\lambda \mid N(t)=n)$, the density of λ conditional on $N(t)=n$.

d) Find $E[\Lambda \mid N(t)=n]$ and interpret your result for very small t with $n = 0$ and for very large t with n large.

e) Find $E[\Lambda \mid N(t)=n, S_1, S_2, \ldots, S_n]$. Hint: Consider the distribution of $S_1 \ldots S_n$ conditional on $N(t)$ and Λ. Find $E[\Lambda \mid N(t)=n, N(\tau)=m]$ for some $\tau < t$.

2.15) a) Use Eq. (31) of chapter 2 to find $E[S_i \mid N(t) = n]$. Hint: In integrating $x\, f_{S_i}(x \mid N(t) = n)$, compare this integral with $f_{S_{i+1}}(x \mid N(t) = n+1)$ and use the fact that the latter expression is a probability density.

b) Find the second moment and the variance of S_i conditional on $N(t) = n$. Hint: Extend the previous hint.

c) Assume that n is odd, and consider $i=(n+1)/2$. What is the relationship between S_i, conditional on $N(t) = n$, and the sample median of n IID uniform random variables.

d) Give a weak law of large numbers for the above median.

2.16) Suppose cars enter a one-way infinite highway at a Poisson rate λ. The ith car to enter chooses a velocity V_i and travels at this velocity. Assume that the V_i's are independent positive random variables having a common distribution F. Derive the distribution of the number of cars that are located in the interval $(0,a)$ at time t.

2.17) Consider an $M/G/\infty$ queue, i.e., a queue with Poisson arrivals of rate λ in which each arrival i, independent of other arrivals, remains in the system for a time X_i, where $\{X_i; i\geq 1\}$ is a set of IID random variables with some given distribution function $F(x)$.

You may assume that the number of arrivals in any interval $(t, t+\varepsilon)$ that are still in the system at some later time $\tau \geq t+\varepsilon$ is *statistically independent* of the number of arrivals in that same interval $(t, t+\varepsilon)$ that have departed from the system by time τ.

a) Let $N(\tau)$ be the number of customers in the system at time τ. Find the mean, $m(\tau)$, of $N(\tau)$ and find $P(N(\tau) = n)$.

b) Let $D(\tau)$ be the number of customers that have departed from the system by time τ. Find the mean, $E[D(\tau)]$, and find $P(D(\tau) = d)$.

c) Find $P(N(\tau)= n, D(\tau) = d)$.

d) Let $A(\tau)$ be the total number of arrivals up to time τ. Find $P(N(\tau) = n \mid A(\tau) = a)$.

e) Find $P(D(\tau+\varepsilon)-D(\tau) = d)$.

2.18) The voters in a given town arrive at the place of voting according to a Poisson process of rate $\lambda = 100$ voters per hour. The voters independently vote for candidate A and candidate B each with probability 1/2. Assume that the voting starts at time 0 and continues indefinitely.

a) Conditional on 1000 voters arriving during the first 10 hours of voting, find the probability that candidate A receives n of those votes.

b) Again conditional on 1000 voters during the first 10 hours, find the probability that candidate A receives n votes in the first 4 hours of voting.

c) Let T be the epoch of the arrival of the first voter voting for candidate A. Find the density of T.

d) Find the PMF of the number of voters for candidate B who arrive before the first voter for A.

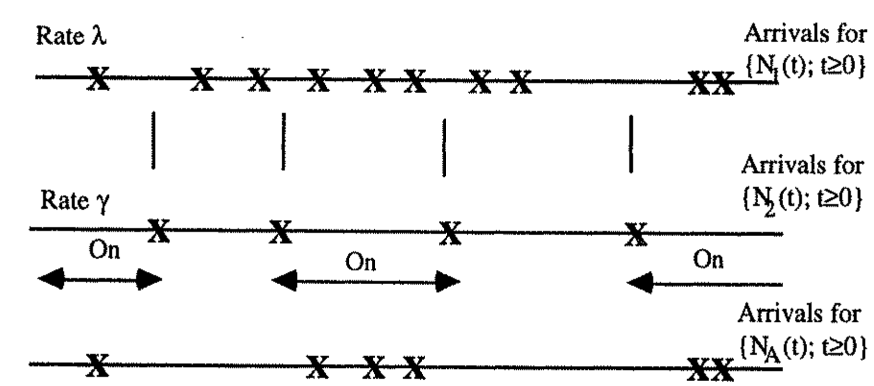
e) Define the n^{th} voter as *a reversal* if the n^{th} voter votes for a different candidate than the n-1[st]. For example, in the sequence of votes A A B A A B B, the third, fourth, and sixth voters are reversals; the third and sixth are A to B reversals and the fourth is a B to A reversal. Let $N(t)$ be the number of reversals up to time t (t in hours). Is $\{N(t); t\geq 0\}$ a renewal process? Is it a delayed renewal process? Explain.

f) Find the expected time (in hours) between reversals.

g) Find the probability density of the time between reversals.

h) Find the density of the time from one A to B reversal to the next A to B reversal.

2.19) Let $\{N_1(t); t \geq 0\}$ be a Poisson counting process of rate λ. Assume that the arrivals from this process are switched on and off by arrivals from a second independent Poisson process $\{N_2(t); t \geq 0\}$ of rate γ.



Let $\{N_A(t); t \geq 0\}$ be the switched process; that is $N_A(t)$ includes the arrivals from $\{N_1(t); t\geq 0\}$ during periods when $N_2(t)$ is even and excludes the arrivals from $\{N_1(t); t\geq 0\}$ while $N_2(t)$ is odd.

a) Find the PMF for the number of arrivals of the first process, $\{N_1(t); t \geq 0\}$, during the n^{th} period when the switch is on.

b) Given that the first arrival for the second process occurs at epoch τ, find the conditional PMF for the number of arrivals of the first process up to τ.

c) Given that the number of arrivals of the first process, up to the first arrival for the second process, is n, find the density for the epoch of the first arrival from the second process.

d) Find the density of the interarrival time for $\{N_A(t); t \geq 0\}$.

2.20) Let us model the chess tournament between Fisher and Spassky as a stochastic process. Let X_i, $i \geq 1$, be the duration of the i^{th} game and assume that $\{X_i; i \geq 1\}$ is a set of IID exponentially distributed rv's each with density $f(x) = \lambda e^{-\lambda x}$. Suppose that each game (independently of all other games, and independently of the length of the games) is won by Fisher with probability p, by Spassky with probability q, and is a draw with probability 1-p-q. The first player to win n games is defined to be the winner, but we consider the match up to the point of winning as being embedded in an unending sequence of games.

a) Find the distribution of time, from the beginning of the match, until the completion of the first game that is won (i.e., that is not a draw). Characterize the process of the number $\{N(t); t \geq 0\}$ of games won up to and including time t. Characterize the process of the number $\{N_F(t); t \geq 0\}$ of games won by Fisher and the number $\{N_S(t); t \geq 0\}$ won by Spassky.

b) For the remainder of the problem, assume that the probability of a draw is zero; i.e., that p+q=1. How many of the first 2n-1 games must be won by Fisher in order to win the match?

c) What is the probability that Fisher wins the match? Your answer should not involve any integrals. Hint: Consider the unending sequence of games and use part (b).

d) Let T be the epoch at which the match is completed (i.e., either Fisher or Spassky wins). Find the distribution function of T.

e) Find the probability that Fisher wins and that T lies in the interval $(t, t+\delta)$ for arbitrarily small δ.

2.21) Using (30), find the conditional density of S_{i+1}, conditional on N(t)=n and $S_i = s_i$ and use this to find the joint density of $S_1, \ldots, S_n$ conditional on N(t)=n. Verify that your answer agrees with (24).

2.22) A two-dimensional Poisson process is a process of randomly occurring special points in the plane such that (i) for any region of area A the number of special points in that region has a Poisson distribution with mean λA, and (ii) the number of special points in nonoverlapping regions is independent. For such a process consider an arbitrary location in the plane and let X denote its distance from its nearest special point (where distance is measured in the usual Euclidean manner). Show that

a) $P(X > t) = \exp(-\lambda \pi t^2)$

b) $E[X] = 1/(2\sqrt{\lambda})$.

2.23) This problem is intended to show that one can analyze the long term behavior of queueing problems by using just notions of means and variances, but that such analysis is awkward, justifying understanding the strong law of large numbers. Consider an M/G/1 queue. The arrival process is Poisson with $\lambda = 1$. The expected service time, E[Y], is 1/2 and the variance of the service time is given to be 1.

a) Consider S_n, the time of the nth arrival, for $n = 10^{12}$. With high probability, S_n will lie within 3 standard derivations of its mean. Find and compare this mean and the 3σ range.

b) Let V_n be the total amount of time during which the server is busy with these n arrivals (i.e., the sum of 10^{12} service times). Find the mean and 3σ range of V_n.

c) Find the mean and 3σ range of I_n, the total amount of time the server is idle up until S_n (take I_n as $S_n - V_n$, thus ignoring any service time after S_n).

d) An idle period starts when the server completes a service and there are no waiting arrivals; it ends on the next arrival. Find the mean and variance of an idle period. Are successive idle periods IID?

e) Combine (c) and (d) to estimate the total number of idle periods up to time S_n. Use this to estimate the total number of busy periods.

f) Combine (e) and (b) to estimate the expected length of a busy period.

NOTES

1. With this density, $P(X_i=0) = 0$, so that we regard X_i as a positive random variable. Since events of probability zero can be ignored, the density $\lambda\exp(-\lambda x)$ for $x \geq 0$ and zero for $x<0$ is effecively the same as the density $\lambda\exp(-\lambda x)$ for $x>0$ and zero for $x \leq 0$.

2. Two processes $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are said to be independent if for all positive integers k and all sets of times $t_1, \ldots, t_k$, the random variables $N_1(t_1), \ldots, N_1(t_k)$ are independent of $N_2(t_1), \ldots, N_2(t_k)$. Here it is enough to extend the independent increment property to independence between increments over the two processes; equivalently, one can require the inter-arrival intervals for one process to be independent of the inter-arrival intervals for the other process.

3. We assume that $\lambda(t)$ is right continuous, i.e., that for each t, $\lambda(t)$ is the limit of $\lambda(t+\varepsilon)$ as ε approaches 0 from above. This allows $\lambda(t)$ to contain discontinuities, as shown in figure 2.6, but follows the convention that the value of the function at the discontinuity is the limiting value from the right. This convention is required in (15) to talk about the distribution of arrivals just to the right of time t.