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3.8 SUMMARY

Sections 1 to 3 give the central results about renewal processes that form the basis for many of the subsequent chapters. The chapter starts with the strong law for renewal processes, showing that the time average rate of renewals, N(t)/t, approaches $1/\overline{X}$ with probability 1 as $t\to\infty$. This, combined with the strong law of large numbers in Chapter 1, is the basis for most subsequent results about time averages. The next topic is the expected renewal rate, E[N(t)]/t. If the Laplace transform of the inter-renewal density is rational, E[N(t)]/t can be easily calculated. In general, the Wald equality shows that $\lim_{t\to\infty} E[N(t)]/t = 1/\overline{X}$. Finally, Blackwell's theorem shows that the renewal epochs reach a steady state as $t\to\infty$. The form of this steady state depends on whether the inter-renewal distribution is arithmetic (see (16)) or non-arithmetic (see (15) and (17)).

Sections 4 and 5 add a reward function R(t) to the underlying renewal process; R(t) depends only on the inter-renewal interval containing t. The time average value of reward exists with probability 1 and is equal to the expected reward over a renewal interval divided by the expected length of an inter-renewal interval. Under some minor restrictions imposed by the key renewal theorem, we also found that, for non-arithmetic inter-renewal distributions, $\lim_{t\to\infty} E[R(t)]$ is the same as the time average value of reward. These general results were applied to residual life, age, and duration, and were also used to derive and understand Little's theorem and the Pollaczek–Khinchin expression for the expected delay in an M/G/1 queue. Finally, all the results above were shown to apply to delayed renewal processes.

For further reading on renewal processes, see [Fel66], [Ros83], or [Wol89]. Feller still appears to be the best source for deep understanding of renewal processes, but Ross and Wolff are somewhat more accessible.

EXERCISES

3.1) Let $\{X_i; i\geq 1\}$ be the inter-renewal intervals of a renewal process generalized to allow for inter-renewal intervals of size 0 and let $P(X_i = 0) = \alpha > 0$. Let $\{Y_i; i\geq 1\}$ be the sequence of *non-zero* interarrival intervals. That is, if $X_1=x_1>0$, $X_2=0$, $X_3=x_3>0$, ..., then $Y_1=x_1$, $Y_2=x_3$, ...

a) Find the distribution function of each Y_i in terms of that of each X_i.

b) Find the PMF of the number of arrivals of the generalized renewal process at each epoch at which arrivals occur.

c) Explain how to view the generalized renewal process as an ordinary renewal process with inter-renewal intervals $\{Y_i; i\geq 1\}$ and bulk arrivals at each renewal epoch.

3.2) Let $\{X_i; i\geq 1\}$ be the inter-renewal intervals of a renewal process and assume that $E[X_i] = \infty$. Let b>0 be an arbitrary number and $\tilde{X}_i$ be a truncated random variable defined by $\tilde{X}_i = X_i$ if $X_i\leq b$ and $\tilde{X}_i = b$ otherwise.

a) Show that for any constant M>0, there is a b sufficiently large so that $E[\tilde{X}_i] \geq M$.

b) Let $\{\tilde{N}(t); t\geq 0\}$ be the renewal process with inter-renewal intervals $\{\tilde{X}_i; i\geq 1\}$ and show that for all t>0, $\tilde{N}(t) \geq N(t)$.

c) Show that for all sample functions n(t), except a set of probability 0, n(t)/t < 2/M for all sufficiently large t. Note: Since M is arbitrary, this means that lim N(t)/t = 0 with probability 1.

3.3) a) Let N be a stopping rule and I_n be the indicator random variable of the event $\{N\geq n\}$. Show that $N = \sum_{n\geq 1} I_n$.

b) Show that $I_1 \geq I_2 \geq I_3 \geq \ldots$.

3.4) Is it true for a renewal process that:

a) N(t) < n if and only if $S_n > t$?

b) N(t) ≤ n if and only if $S_n \geq t$?

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3.5) Let $\{N(t); t \geq 0\}$ be a renewal process and let m(t) = E[N(t)] be the expected number of arrivals up to and including time t. Let $\{X_i; i \geq 1\}$ be the inter-renewal times and assume that $F_X(0) = 0$.

a) Show that $E[N(t) \mid X_1 = x] = E[N(t-x)] + 1$ for x < t.

b) Use part (a) to show that $m(t) = F_X(t) + \int_0^t m(t-x)dF_X(x)$ for t>0, This equation is known as the renewal equation and an alternative derivation is given in Eq. (5).

c) Suppose that X is an exponential random variable of parameter λ. Evaluate $L_m(s)$ from (6); verify that the inverse Laplace transform is λt; $t \geq 0$.

3.6) a) Let the inter-renewal interval of a renewal process have a second order Erlang density, $f_X(x) = \lambda^2 x \exp(-\lambda x)$. Evaluate the Laplace transform of m(t) = E[N(t)].

b) Use this to evaluate m(t) for t≥0. Verify that your answer agrees with (8).

c) Evaluate the slope of m(t) at t=0 and explain why that slope is not surprising.

d) View the renewals here as being the even numbered arrivals in a Poisson process of rate λ. Sketch m(t) for the process here and show one half the expected number of arrivals for the Poisson process on the same sketch. Explain the difference between the two.

3.7) a) Let N(t) be the arrivals in the interval (0,t] for a Poisson process of rate λ. Show that the probability that N(t) is even is $[1+ \exp(-2\lambda t)]/2$. Hint: Look at the power series expansion of $\exp(-\lambda t)$ and that of $\exp(\lambda t)$, and look at the sum of the two. Compare this with $\sum_{n \text{ even}} P(N(t) = n)$.

b) Let $\tilde{N}(t)$ be the number of even numbered arrivals in (0,t]. Show that $\tilde{N}(t) = N(t)/2 - I_{odd}(t)/2$ where $I_{odd}(t)$ is a random variable that is 1 if N(t) is odd and 0 otherwise.

c) Use parts (a) and (b) to find $E(\tilde{N}(t))$. Note that this is m(t) for a renewal process with 2nd order Erlang inter-renewal intervals.

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c) Use parts (a) and (b) to find $E(\breve{N}(t))$. Note that this is $m(t)$ for a renewal process with 2nd order Erlang inter-renewal intervals.

3.8) Use Wald's equality to compute the expected number of trials of a Bernoulli process up to and including the k^{th} success.

3.9) A gambler starts to play a dollar slot machine with an initial finite capital of d>0 dollars. At each play, either his dollar is lost or is returned with some additional number of dollars. Let X_i be his change of capital on the i^{th} play. Assume that $\{X_i; i = 1, 2, ...\}$ is a set of IID random variables taking on integer values $\{-1, 0, 1, ...\}$. Assume that $E[X_i] < 0$. The gambler plays until losing all his money (i.e., the initial d dollars plus subsequent winnings).

a) Let N be the number of plays until the gambler loses all his money. Is the weak law of large numbers sufficient to argue that $\lim_{n\to\infty} P(N > n) = 0$ (i.e., that N is a random variable) or is the strong law necessary?

b) Find E[N].

3.10) Let $\{X_i; i \geq 1\}$ be IID binary random variables with $P_X(0) = P_X(1) = 1/2$. Let N be a non negative integer valued random variable defined on the above sample space of binary sequences. Find the simplest example you can in which N is *not* a stopping rule for $\{X_i; i \geq 1\}$ and where $E[X]E[N] \neq E[S_N]$ where $S_N = \sum_{i=1}^{N} X_i$.

3.11) Let $N = \min\{n \mid S_n \leq B \text{ or } S_n \geq A\}$, where A is a positive integer, B is a negative integer, and $S_n = X_1+X_2+...+X_n$. Assume that $\{X_i; i\geq 1\}$ is a set of *zero mean* IID rv's that can take on only the set of values $\{-1, 0, \text{or } +1\}$, each with positive probability.

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3.12) Let $\{N(t); t \geq 0\}$ be a renewal process generalized to allow for inter-renewal intervals $\{X_i\}$ of duration 0. Let each X_i have the PMF $P(X_i = 0) = 1-\varepsilon$; $P(X_i = 1/\varepsilon) = \varepsilon$.

a) Sketch a typical sample function of $\{N(t); t \geq 0\}$. Note that N(0) can be non-zero (i.e., N(0) is the number of zero interarrival times that occur before the first non-zero interarrival time).

b) Evaluate E[N(t)] as a function of t.

c) Sketch E[N(t)]/t as a function of t.

d) Evaluate $E[S_{N(t)+1}]$ as a function of t (do this directly, and then use Wald's equality as a check on your work).

e) Sketch the lower bound $E[N(t)]/t \geq 1/E[X] - 1/t$ on the same graph with part (c).

f) Sketch $E[S_{N(t)+1} - t]$ as a function of t and find the time average of this quantity.

g) Evaluate $E\{S_{N(t)}]$ as a function of t; verify that $E[S_{N(t)}] \neq E[X]E[N(t)]$.

3.13) Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two-day's travel; door 2 returns him to his room after four-day's travel; and door 3 returns him to his room after eight-day's travel. Suppose each door is equally likely to be chosen whenever he is in the room, and let T denote the time it takes the miner to become free.

a) Define a sequence of independent and identically distributed random variables $X_1, X_2, ...$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i$$

Note: You may imagine that the miner continues to randomly choose doors even after he reaches safety.

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3.14) Let $Y(t) = S_{N(t)+1} - t$ be the residual life at time t of a renewal process. *First* consider a renewal process in which the interarrival time has density $f_X(x) = e^{-x}$; $x \geq 0$, and *next* consider a renewal process with density

$$f_X(x) = \frac{3}{(x+1)^4} \;;\; x\geq 0$$

For *each* of the above densities, use renewal reward theory to find:

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ii) the second moment in time of Y(t) (i.e., $\lim_{T\to\infty} \frac{1}{T} \int_0^T Y^2(t)\, dt$)

For the exponential density, verify your answers by finding E[Y(t)] and $E[Y^2(t)]$ directly.

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Successive service times (for those customers that are served) are IID and independent of arrival times. Assume that the first customer arrives and enters service at time t=0.

a) Show that the sequence of times $S_1, S_2, ...$ at which new customers enter service are the renewal times of a renewal process. Show that each inter-renewal interval $X_i = S_i - S_{i-1}$ (where $S_0 = 0$) is the sum of two independent random variables, $Y_i + U_i$ where Y_i is the i^{th} service time; find the probability density of U_i.

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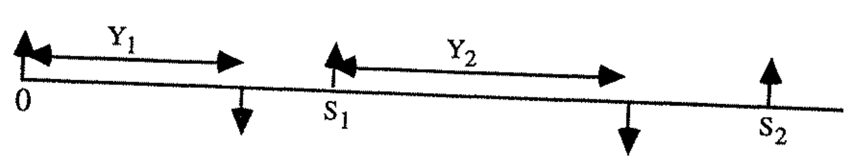
b) Assume that a reward (actually a cost in this case) of one unit is incurred for each customer turned away. Sketch the expected reward function as a function of time for the given sample below of inter-renewal intervals and service intervals; the expectation is to be taken over those (unshown) arrivals of customers that must be turned away.

c) Let $\int_0^t R(\tau)d\tau$ denote the accumulated reward (i.e., cost) from 0 to t and find the limit as $t \to \infty$ of $\int_0^t R(\tau)d\tau$. Explain (without any attempt to be rigorous or formal) why this limit exists with probability 1.



d) In the limit of large t, find the expected reward from time t until the next renewal. Hint: Sketch this expected reward as a function of t for a given sample of inter-renewal intervals and service intervals; then find the time average.

e) Now assume that the arrivals are deterministic, with the first arrival at time 0 and the n^{th} arrival at time n-1. Does the sequence of times S_1, S_2, ... at which new customers start service still constitute the renewal times of a renewal process? Draw a sketch of arrivals, departures, and service time intervals. Again find $\lim_{t\to\infty}(\int_0^t R(\tau)d\tau)/t$

3.16) Let $Z(t) = t - S_{N(t)}$ be the age of a renewal process and $Y(t) = S_{N(t)+1} - t$ be the residual life. Let $F_X(x)$ be the distribution function of the inter-renewal interval and find the following as a function of $F_X(x)$:

a) $P(Y(t) > x \mid Z(t) = s)$

b) $P(Y(t) > x \mid Z(t + x/2) = s)$

c) $P(Y(t) > x \mid Z(t+x) > s)$ for a Poisson process.

3.17) Let Z(t), Y(t), X(t) denote the age, residual life, and duration at time t for a renewal process $\{N(t); t\geq 0\}$ in which the interarrival time has a density given by f(x). Find the following probability densities; assume steady state.

a) $f_{Y(t)}(y \mid Z(t+s/2)=s)$ for given s>0.

b) $f_{Y(t),Z(t)}(y,z)$.

c) $f_{Y(t)}(y \mid X(t)=x)$.

d) $f_{Z(t)}(z \mid Y(t-s/2)=s)$ for given s>0.

e) $f_{Y(t)}(y \mid Z(t+s/2)\geq s)$ for given s>0.

3.18) a) Find $\lim_{t\to\infty}\{E[N(t)] - t/E[X]\}$ for a renewal process $\{N(t); t \geq 0\}$ with inter-arrival times $\{X_i; i \geq 1\}$. Hint: Use Wald's equation.

b) Evaluate your result for the case in which X is an exponential random variable (you already know what the result should be in this case).

c) Evaluate your result for a case in which $E[X] < \infty$, $E[X^2] = \infty$. Explain (very briefly) why this does not contradict the elementary renewal theorem.

3.19) Customers arrive at a bus stop according to a Poisson process of rate λ. Buses arrive according to a renewal process with the inter-renewal interval distribution $F_X(x)$. At the epoch of a bus arrival, all waiting passengers enter the bus and the bus leaves immediately. Let R(t) be the number of customers waiting at time t.

a) Draw a sketch of a sample function R(t).

b) Given that the first bus arrives at time x (i.e. $X_1 = x$), find the expected number of customers picked up; then find $E\left[\int_0^x R(t)dt\right]$, again given the first bus arrival at x.

c) Find $\lim_{t\to\infty} \frac{1}{t}\int_0^t R(\tau)d\tau$ (with probability 1). Assuming that F_X is a non-arithmetic distribution, find $\lim_{t\to\infty} E[R(t)]$. Interpret what these quantities mean.

d) Find the time average expected wait per customer.

e) Find the fraction of time that there are no customers at the bus stop. (Hint: This part is independent of (a), (b), and (c); check your answer for $E[X] << 1/\lambda$).

3.20) Let $\{N_1(t); t \geq 0\}$ be a Poisson counting process of rate λ. Assume that the arrivals from this process are switched on and off by arrivals from a non-arithmetic renewal process $\{N_2(t); t \geq 0\}$. The two processes are independent.

Let $\{N_A(t); t \geq 0\}$ be the switched process; that is $N_A(t)$ includes arrivals from $\{N_1(t); t \geq 0\}$ while $N_2(t)$ is even and excludes arrivals from $\{N_1(t); t \geq 0\}$ while $N_2(t)$ is odd.

a) Is $N_A(t)$ a renewal process? Explain your answer and if you are not sure, look at several examples for $N_2(t)$.

b) Find $\lim_{t\to\infty} \frac{1}{t}N_A(t)$ and explain why the limit exists with probability 1. Hint: Use symmetry—that is, look at $N_1(t) - N_A(t)$. To show why the limit exists, use the renewal reward theorem. What is the appropriate renewal process to use here?