Renewal Processes/Counting Processes/Poisson Processes

**Renewal Process:** Interarrival intervals are positive IID random variables. (Renewal processes are more general than it might seem)

\[ N(t): \text{number of arrivals in } (0,t] \]

\[ S_n: \text{Epoch of } n \text{th arrival} \]

\[ \{X_1, X_2, X_3, \ldots\}: \text{interarrival intervals} \]

\[ X_1 = S_1, \quad X_n = S_n - S_{n-1}, \quad S_n = \sum_{i=1}^{n} X_i \]

{Counting Processes}

\( \{N(t), t \geq 0\} \): family of random variables

\( N(t): \text{number of arrivals in interval } (0,t] \).

\( N(0) = 0 \) with probability 1

**Counting Process** \( \{N(t), t \geq 0\} \): family of non-negative integer valued random variables (one for each \( t \geq 0 \)) with the properties that

\[ N(\tau) \geq N(t), \quad \tau \geq t \]

and \( N(0) = 0 \) with probability 1.

Equivalent: \( \{S_1, S_2, \ldots\} \) or \( \{X_1, X_2, \ldots\} \) or \( \{N(t), t \geq 0\} \).

**Note:** \( \{S_n \leq t\} = \{N(t) \geq n\} \)

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**Poisson Process**

- Renewal process with \( F_X(x) = 1 - e^{-\lambda x} \)
- \( \lambda t \): expected number of arrivals in interval of length \( t \)
- \( E[N(t)] = \lambda t \)
- Memoryless Property
  \[ P(X > t + x \mid X > t) = P(X > x), \quad x \geq 0 \]
- \( P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \)

**Theorem** (informal statement):
(a) Interarrival interval from \( t \) until the first arrival after \( t \) is a R.V. with \( F_X(x) = 1 - e^{-\lambda x} \)
(b) This R.V. is independent of all arrival epochs before time \( t \) and \( N(\tau) \) for \( \tau \leq t \)

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**Stationary Increment Property**

\( \{N(t), t \geq 0\} \): counting process

\( \tilde{N}(t, t') = N(t') - N(t): \text{number of arrivals in } (t, t'], t' \geq t \)

\( \tilde{N}(t, t') \) has same distribution as \( N(t' - t) \)
Independent Increment Property

\[ \{N(t_1), \tilde{N}(t_2, t_1), \tilde{N}(t_3, t_2)\} \text{ independent random variables} \]

Definition Poisson Process

**Definition 1:** Renewal process with \( F_X(x) = 1 - e^{-\lambda x} \)

**Definition 2:** Counting process \( \{N(t), t \geq 0\} \) with independent and stationary increment properties, and

\[ P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \]

**Definition 3:** Counting process \( \{N(t), t \geq 0\} \) with independent and stationary increment properties, and

\[ \tilde{N}(t + \delta, t) = 0 = 1 - \lambda \delta + o(\delta) \]
\[ \tilde{N}(t + \delta, t) = 1 = \lambda \delta + o(\delta) \]
\[ \tilde{N}(t + \delta, t) \geq 2 = o(\delta) \]

Combining Independent Poisson Processes

Bernoulli Splitting of Poisson Processes

The two processes are independent