The front is a binary image of pixels to be filled next. Imagine the following fill front image \( I(x,y) \in [0,1] \) where:

\[
I(x,y) = 0 \quad \text{when} \quad \square
\]

\[
I(x,y) = 1 \quad \text{when} \quad \blacktriangleleft \quad (x,y)
\]

Assume we are estimating the normal \( \mathbf{n}(x,y) \) at pixel \((x,y)\) marked in red. A second degree curve fit to the data would look like:

For convenience, assume the origin is now at \((x,y)\) so that:

This means that the x coordinates of the front line are:

\[
X = [-3, -2, -1, 0, 1, 2, 3, 4, 5] \quad \text{and for } y \text{ they are:}
\]

\[
Y = [-4, -3, -2, -1, 0, 1, 1, 1, 2, 3]
\]

If we use a window of width \( w = 3 \) we are left with:

\[
X = [-2, -1, 0, 1, 2, 3]
\]

\[
Y = [-3, -2, -1, 0, 1, 1, 1, 1] \]
Copied here for convenience:

\[ X = [-2, -1, 1, 0, 1, 2, 3] \]

\[ Y = [-3, -2, 1, 0, 1, 1, 1] \]

We now fit 2nd order polynomials to each of these two vectors. To obtain the approximations to the coordinate functions:

\[ Y(t) = [X(t), Y(t)] \]

COORDINATE FUNCTIONS

For instance, the vector \( X(t) \) has the constraints:

\[
\begin{align*}
X(-3) &= 2 \\
X(-2) &= 2 \\
X(-1) &= 1 \\
X(0) &= 0 \\
X(1) &= 1 \\
X(2) &= 3
\end{align*}
\]

In these constraints, we are used to estimate the parameters of a 2nd degree polynomial:

\[ X(t) = at^2 + bt + c. \]

The least squares solution defines the system:

\[
\begin{bmatrix}
1 & -3 & 9/2 \\
1 & -2 & 2 \\
1 & -1 & 1/2 \\
1 & 0 & 0 \\
1 & 1 & 1/2 \\
1 & 2 & 2 \\
1 & 3 & 9/2
\end{bmatrix}
\begin{bmatrix}
X(0) \\
X'(0) \\
X''(0)
\end{bmatrix} =
\begin{bmatrix}
-2 \\
-2 \\
-1 \\
0 \\
1 \\
3
\end{bmatrix}
\]

Use the above inverse matrix to solve for \( X(0), X'(0), X''(0) \). Repeat for \( Y(t) \) to get \( Y(0), Y'(0), Y''(0) \).

To get the normal, take the first derivative \( \frac{dX(t)}{dt}, \frac{dY(t)}{dt} \) for each function and use:

\[ N(t) = \frac{1}{|| \left( \frac{dX(t)}{dt}, \frac{dY(t)}{dt} \right) ||} \left( \frac{dX(t)}{dt}, \frac{dY(t)}{dt} \right) \text{ at } t=0. \]
To incorporate the Gaussian weights, you must find the roots of the each congruent, so in the form:

\[
\begin{align*}
L &= 3 \\
& \quad \begin{bmatrix} -3 & 9/2 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} X(1) \\ \frac{dX(1)}{dt} \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\
& \quad \begin{bmatrix} L = 2 \\
& \quad \begin{bmatrix} -1 & 1/2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} X(2) \\ \frac{dX(2)}{dt} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
& \quad \begin{bmatrix} L = 3 \\
& \quad \begin{bmatrix} 3 & 9/2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} X(3) \\ X(3) \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}
\end{align*}
\]

The first row of the Gaussian weighted matrix is:

\[
\begin{bmatrix}
\omega(3)^{1/2} & \omega(3)^{1/2} \\
\omega(3)^{1/2} & \omega(3)^{1/2} \\
\omega(3)^{1/2} & \omega(3)^{1/2}
\end{bmatrix}
\begin{bmatrix}
X(0) \\
X(0)
\end{bmatrix}
\begin{bmatrix}
\omega(3)^{1/2} \\
\omega(3)^{1/2} \\
\omega(3)^{1/2}
\end{bmatrix}
\begin{bmatrix}
X(0) \\
X(0)
\end{bmatrix}
\]

Then you may use \( X'(0) \) and \( V'(0) \) to estimate the unit normal.