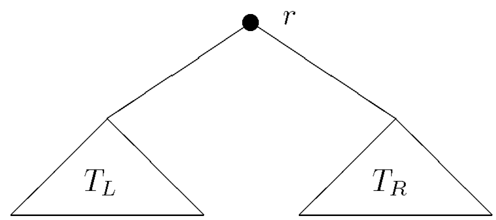
Inductive Definition of Some Sets of Binary Trees

In the following definitions, the left subtree $T_L = (V_L, E_L)$ and right subtree $T_R = (V_R, E_R)$ are assumed to be node disjoint (i.e. $V_L \cap V_R = \emptyset$) with roots r_L and r_R. Moreover,



$r \notin V_L \cup V_R$ and (in order to formally distinguish the left and right subtrees) the edge (r, r_L) is labelled "LEFT" and the edge (r, r_R) is labelled "RIGHT". To simplify the definitions, we define the empty tree as a tree of height -1. We will define binary trees, full binary trees, perfectly complete binary trees and complete binary trees by induction, using the same base cases.

Base cases: **i)** The empty tree is a tree of height -1.

ii) The directed graph $G = (V, E)$ with $V = \{r\}$ and $E = \emptyset$ is a (full, perfectly complete, complete) binary tree of height 0.

Induction step for each definition. Let $h > 0$.

A) A binary tree of height h is a (edge labelled) directed graph $G = (V, E)$ with left subtree T_L and right subtree T_R where $V = \{r\} \cup V_L \cup V_R$ and $E = \{(r, r_L), (r, r_R)\} \cup E_L \cup E_R$, $T_L = (V_L, E_L)$ and $T_R = (V_R, E_R)$ are binary trees of height i $(-1 \leq i < h)$ and either T_L or T_R (or both) are of height $h - 1$.

Note: If T_L (respectively, T_R) is of height -1, then this subtree is missing.

B) A full binary tree of height h is defined exactly as a binary tree except that T_L and T_R must have height ≥ 0 (i.e. can't be empty).

C) A perfectly complete binary tree of height h is a directed graph $G = (V, E)$ where $V = \{r\} \cup V_L \cup V_R$ and $T_L = (V_L, E_L)$ and $T_R = (V_R, E_R)$ are perfectly complete binary trees of height $h - 1$.

D) A complete tree of length h is a directed graph $G = (V, E)$ with left subtree T_L and right tree T_R where either

1. T_R is a perfectly complete binary tree of height $h - 2$ and T_L is a complete binary tree of height $h - 1$,

   or

2. T_L is a perfectly complete binary tree of height $h - 1$ and T_R is a complete binary tree of height $h - 1$.

Note that a perfectly complete binary tree is certainly a complete binary tree (using this definition).