

Care and feeding of a proof

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I'll be marking homework assignments for B38, so it is fair that I let you know in advance my tastes (and distastes) in the difficult art of expressing a proof.

I agree with Donald Knuth that the source code of a computer programme is primarily a literary work (meant to communicate with other programmers) and the fact that it compiles into a (possibly) useful programme is a positive side-effect.

Similarly, a proof is primarily a piece of creative work meant to convince others of some claim. A really good proof will do this clearly, briefly, and with an elegance that excites the reader about both the main claim and the methods used to prove it. The fact that a correct proof might result in your getting a good mark, a publication in a scholarly publication, or advancing your career is just a side-effect (although probably a pleasant one).

You might wonder how *any* proof could satisfy all these (possibly competing) criteria. In practice every proof I've ever seen falls somewhat short of one of the above criteria — it fails on brevity, clarity, or elegance, even when it passes the test of correctness (that is, it convinces the community reading it that the claim has been shown to be true). I hope that you use the time and resources you have to present the best possible proofs in your assignments.

Do this

Here are some of the things I think you should strive for in your proofs:

Be convincing You have to present an argument that first convinces yourself. If you are discussing your work with another student, you might ask them to listen to (not read! — recall the policy on collaboration) a sketch of your proof, trying to find weak points. Once you have a proof that convinces you and a “friendly” audience, try to imagine the criticism that a really picky audience might make of your proof, and fix the parts they might criticize.

Be brief but clear The person reading your proof has a finite amount of time and attention during which to grasp it. You should anticipate how much time and attention your audience can give you, and edit your proof accordingly.

Brevity should not be at the expense of clarity. Any step in your proof that doesn't seem obvious has to be explained. You have the difficult task of deciding which things are obvious (and can be omitted). What is obvious to one audience may not be obvious to another.

Be imaginative There is often more than one way to prove a given claim. Two (or more) independent proofs of the same claim increase our understanding of why it is true: in a sense they make it “truer.” If you think of an unusual approach to a proof, try to turn it into a precise, convincing proof instead of, or (if you want to hedge your bets) in addition to, a more standard approach.

Be pragmatic You might get stumped. If so, don’t abandon the proof. You can certainly present a sketch of what steps you lack in order to have a valid proof, and how you would proceed if *only* you had those steps. You might be able to prove something “weaker,” but similar to, the required result. This pragmatic approach may get you some part marks, and it certainly helps focus your thinking on what is missing in your proof.

Don’t do that . . .

Rather than rant about my pet peeves, I will try to turn them into general guidelines (or perhaps anti-guidelines).

Don’t write nonsense If you get stuck on some part of your proof, don’t substitute weak, unconvincing arguments for sound reasoning. Clearly note which things you can prove and which you can’t and move on.

Don’t overuse symbols A certain amount of mathematical notation helps achieve the brevity and clarity I mentioned above. However, if your proof starts to look like the output of a bad modem connection, you should try this test:

Try to replace some symbolic expressions with *precise* English prose. If this doesn’t make the proof impossibly long, ask yourself how it affects the clarity — often it is an improvement. Even if you don’t keep the prose version, this exercise increases your understanding of the proof.

Don’t blindly follow a template If you have seen a particular style or format for a proof somewhere, you are certainly welcome to try to emulate it, where appropriate. But be sure to understand why (or if . . .) each part of your proof contributes to convincing your audience that the claim is true.

Here’s an example. I’ve seen lots of proofs that include a stack of equations something like this (exaggerated) example:

$$\begin{aligned}x + y + z &= n^2/2 \\ f(t) &= 17 \\ \ln(n) &= y \\ 3 &\geq 2\end{aligned}$$

Now, the last inequality is certainly true. But how is it related to the previous equations? Is it implied by them, does it imply them, or both? I certainly don’t know, so if you present a

stack of equations you need to explicitly say how they're related to each other (and the rest of your proof).

Notice how this is very different from:

$$\begin{aligned}x + y + z &= n^2/2 \\ &= f(t) \\ &= 3 \\ &\geq 2\end{aligned}$$

This stack very clearly claims that $x + y + z$ is no less than 2, which may be true or false, but is at least comprehensible.

You might even blindly follow the template of mathematical induction. Don't assume that every proof *needs* this method. I've seen something like the following attempt to prove that it is impossible to choose natural numbers j and k such that $5j + 8k = 27$:

$P(n)$: there are no natural numbers j and k such that $5j + 8k = 27$.

The variable n doesn't even appear in the body of the predicate, so the author seems to be guilty of writing nonsense, as well as blindly following a template. I don't think that induction is really appropriate for this problem, since only a few instances of j or k need to be checked (rather than some infinite subset of the natural numbers).

Don't abuse your friends If you collaborate with 1 or 2 classmates, don't follow their solutions too closely (notice that according to course policy these exchanges should be verbal, not written). You not only run the risk of falling into plagiarism, but you risk stunting your intellectual development.

Good collaboration includes pointing out weaknesses in your friend's proof and occasionally sharing small insights that get you started in the right direction. It doesn't include sacrificing your own creativity in solving (and expressing the solution of) a problem by following somebody else's path.

There are certainly more aspects of what constitutes a good (or bad ...) proof, but I think that's enough to get you started. Good luck.