Computer Science 238 (L5101) Fall 1999
St. George Campus University of Toronto

Answers to Homework Assignment #5

ANSWER TO QUESTION 1.

(a) This is false. Here is a counterexample: Let $R = 0^*$, $S = 0$ and $T = (0 + 00)$. We have,

$$\begin{aligned} \mathcal{L}(RT) &= \{0^{n+1} : n \in \mathbb{N}\} \cup \{0^{n+2} : n \in \mathbb{N}\} \\ &= \{0^{n+1} : n \in \mathbb{N}\} \\ &= \mathcal{L}(RS) \end{aligned}$$

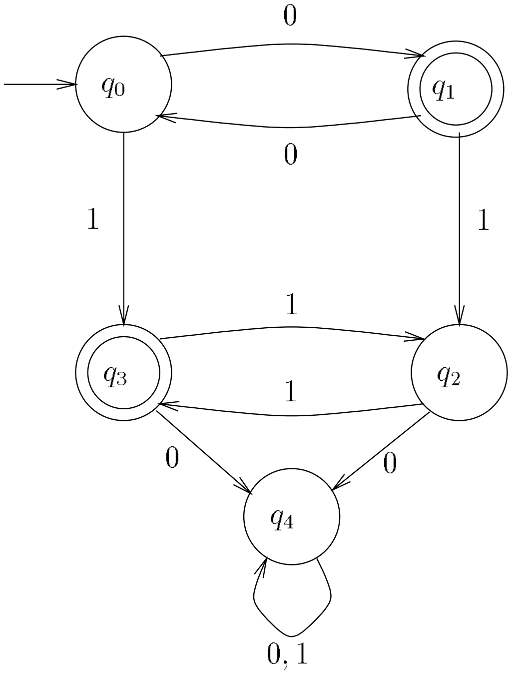
and so $RT \equiv RS$. Also, $R \not\equiv \emptyset$. However, $S \not\equiv T$, (because $\mathcal{L}(S) = \{0\}$ while $\mathcal{L}(T) = \{0, 00\}$).

(b) This is true. First, note that by distributivity of concatenation over union and the fact that $R\epsilon \equiv R$, the left-hand side is equivalent to $\big(R(S+\epsilon)\big)^* R$, and the right-hand side is equivalent to $R\big((S+\epsilon)R\big)^*$, Thus, it is sufficient to prove that $\big(R(S+\epsilon)\big)^* R \equiv R\big((S+\epsilon)R\big)^*$.

To see this, first consider any $x \in \mathcal{L}\big(\big(R(S+\epsilon)\big)^* R\big)$. Thus, $x = x_1 y_1 x_2 y_2 \ldots x_k y_k x_{k+1}$, for some $k \in \mathbb{N}$, where $x_i \in \mathcal{L}(R)$, for all $1 \le i \le k+1$, and $y_i \in \mathcal{L}(S+\epsilon)$, for all $1 \le i \le k$. Thus, $y_i x_{i+1} \in \mathcal{L}\big((S+\epsilon)R\big)$, for all $1 \le i \le k$. This implies that $x = x_1 z$, where $z \in \mathcal{L}\big(\big((S+\epsilon)R\big)^*\big)$. Since $x_1 \in \mathcal{L}(R)$, it follows that $x \in \mathcal{L}\big(R\big((S+\epsilon)R\big)^*\big)$. This proves that $\mathcal{L}\big(\big(R(S+\epsilon)\big)^* R\big) \subseteq \mathcal{L}\big(R\big((S+\epsilon)R\big)^*\big)$. A similar argument shows that the converse is true, so that $\big(R(S+\epsilon)\big)^* R \equiv R\big((S+\epsilon)R\big)^*$.

ANSWER TO QUESTION 2.

(a) Following is a DFSA M that accepts L.



A regular expression that denotes L is $(00)^*1(11)^*+0(00)^*(11)^*$. (Informally, this says: an even number of 0s followed by an odd number of 1s, or an odd number of 0s followed by an even number of 1s.)

To prove that M accepts L we characterise the states of M with state invariants. Let $P(x)$ be the following predicate on strings $x \in \{0,1\}^*$ (where δ is the transition function of M):

$$P(x): \qquad \delta^*(q_0, x) = \begin{cases} q_0, & \text{if } x = 0^k, \text{ for some } k \in \mathbb{N}, \text{ and } |x| \text{ is even} \\ q_1, & \text{if } x = 0^k, \text{ for some } k \in \mathbb{N}, \text{ and } |x| \text{ is odd} \\ q_2, & \text{if } x = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N}, \ \ell > 0, \text{ and } |x| \text{ is even} \\ q_3, & \text{if } x = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N}, \ \ell > 0, \text{ and } |x| \text{ is odd} \\ q_4, & \text{if 10 is a substring of } x \end{cases}$$

We prove that $P(x)$ holds for every $x \in \{0,1\}^*$ using structural induction on x.

Basis: $x = \epsilon$. Then $x = 0^k$, for $k = 0$, and $|x|$ is even. Furthermore, by the definition of δ^*, $\delta^*(q_0, x) = q_0$. Thus, $P(x)$ holds in this case.

Induction Step: $x = ya$, for some $y \in \{0,1\}^*$ and $a \in \{0,1\}$. We assume, by induction, that $P(y)$ holds. That is,

$$\delta^*(q_0, y) = \begin{cases} q_0, & \text{if } y = 0^k, \text{ for some } k \in \mathbb{N}, \text{ and } |y| \text{ is even} \\ q_1, & \text{if } y = 0^k, \text{ for some } k \in \mathbb{N}, \text{ and } |y| \text{ is odd} \\ q_2, & \text{if } y = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N}, \ \ell > 0, \text{ and } |y| \text{ is even} \\ q_3, & \text{if } y = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N}, \ \ell > 0, \text{ and } |y| \text{ is odd} \\ q_4, & \text{if 10 is a substring of } y \end{cases}$$

We will prove that $P(x)$ holds as well. There are two cases

Case 1. $a = 0$. Then $x = y0$. We have

$$\begin{aligned} \delta^*(q_0, x) &= \delta^*(q_0, y0) \\ &= \delta\big(\delta^*(q_0, y), 0\big) \\ &= \begin{cases} \delta(q_0, 0), & \text{if } y = 0^k, \text{ for some } k \in \mathbb{N}, \ |y| \text{ even} \\ \delta(q_1, 0), & \text{if } y = 0^k, \text{ for some } k \in \mathbb{N}, \ |y| \text{ odd} \\ \delta(q_2, 0), & \text{if } y = 0^k1^\ell, \text{ for some } k \in \mathbb{N}, \ \ell > 0, \ |y| \text{ even} \\ \delta(q_3, 0), & \text{if } y = 0^k1^\ell, \text{ for some } k \in \mathbb{N}, \ \ell > 0, \ |y| \text{ odd} \\ \delta(q_4, 0), & \text{if 10 is a substring of } y \end{cases} && \text{[by induction hypothesis]} \\ &= \begin{cases} q_1, & \text{if } x = 0^k, \text{ for some } k \in \mathbb{N}, \ |x| \text{ odd} \\ q_0, & \text{if } x = 0^k, \text{ for some } k \in \mathbb{N}, \ |x| \text{ even} \\ q_4 & \text{if 10 is a substring of } x \end{cases} && \text{[by the transition diagram]} \end{aligned}$$

Case 2. $a = 1$. Then $x = y1$. We have

$$\begin{aligned}
\delta^*(q_1, x) &= \delta^*(q_0, y1) \\
&= \delta\big(\delta^*(q_0, y), 1\big) \\
&= \begin{cases} \delta(q_0, 1), & \text{if } y = 0^k, \text{ for some } k \in \mathbb{N},\ |y| \text{ even} \\ \delta(q_1, 1), & \text{if } y = 0^k, \text{ for some } k \in \mathbb{N},\ |y| \text{ odd} \\ \delta(q_2, 1), & \text{if } y = 0^k1^\ell, \text{ for some } k \in \mathbb{N},\ \ell > 0,\ |y| \text{ even} \\ \delta(q_3, 1), & \text{if } y = 0^k1^\ell, \text{ for some } k \in \mathbb{N},\ \ell > 0,\ |y| \text{ odd} \\ \delta(q_4, 1), & \text{if } 10 \text{ is a substring of } y \end{cases} && \text{[by induction hypothesis]} \\
&= \begin{cases} q_3, & \text{if } x = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N},\ \ell > 0,\ |x| \text{ odd} \\ q_2, & \text{if } x = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N},\ \ell > 0,\ |x| \text{ even} \\ q_3, & \text{if } x = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N},\ \ell > 0,\ |x| \text{ odd} \\ q_2, & \text{if } x = 0^k1^\ell, \text{ for some } k, \ell \in \mathbb{N},\ \ell > 0,\ |x| \text{ even} \\ q_4 & \text{if } 10 \text{ is a substring of } x \end{cases} && \text{[by the transition diagram]}
\end{aligned}$$

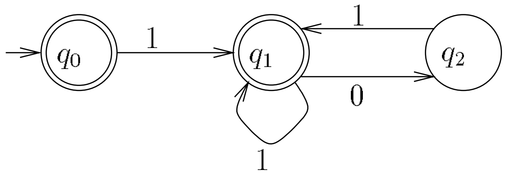
Combining the results of the two cases we get that $P(x)$ holds, as wanted.

This completes the proof that $P(x)$ holds for all strings x. Using this we can now prove that $\mathcal{L}(M) = L$.

$L \subseteq \mathcal{L}(M)$: If $x \in L$, then $x = 0^k1^\ell$, for some $k, \ell \in \mathbb{N}$, and $|x|$ is odd. By $P(x)$, $\delta^*(q_0, x) \in \{q_1, q_3\}$, and since q_1 and q_3 are accepting states, x is accepted by M.

$\mathcal{L}(M) \subseteq L$: If x is accepted by M, then 10 is not a substring of x (otherwise, by $P(x)$, $\delta^*(q_0, x) = q_4$, which is not an accepting state). Therefore $x = 0^k1^\ell$, for some $k, \ell \in \mathbb{N}$. Also, $|x|$ is not even (otherwise, by $P(x)$, $\delta^*(q_0, x) \in \{q_0, q_2\}$, and neither of these is an accepting state). Therefore $|x|$ is odd. Since $x = 0^k1^\ell$, for some $k, \ell \in \mathbb{N}$ and $|x|$ is odd, $x \in L$.

(b) Following is a DFSA M' that accepts L'.



A regular expression that denotes L' is $\epsilon + 1(1^*01)^*1^*$.

To prove that M' accepts L' we first note that L' consists of the set of strings that:

- do not start with 0;
- do not end with 0; and
- do not contain 00 as a substring.

(A string that violates one of these properties contains a 0 that is not surrounded by 1s. Conversely, if x contains a 0 that is not surrounded by 1s, this 0 is the first or last symbol in x, or it is preceded or followed by another 0. In other words x violates one of the above three propreties.)

By inspection of the transition diagram, M' rejects a string x if and only if:

- M' is in q_0 and the next symbol is 0. This means that x starts with 0. (Because there is no transition into q_0, so M' is in this state q_0 *only* at the start.)

- M' is in state q_1, and the last symbol of x is 0.
- At some point M' is in state q_2 and the next symbol is 0. This means that the previous symbol seen by M' was 0 (because the only transition into q_2 is on 0). Therefore x contains 00 as a substring.

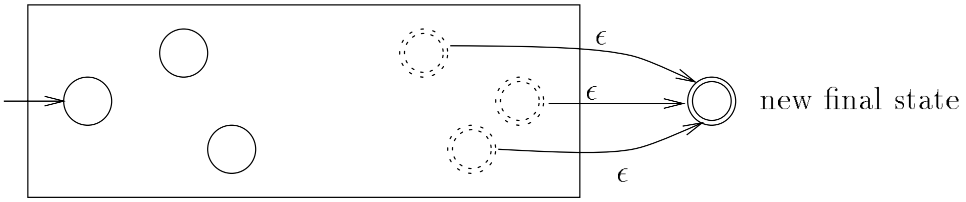
In other words, M' rejects x if and only if x is not in L'. Therefore M' accepts L.

Any string that violates any of these properties is rejected by M. Conversely, a string is rejected by M if M is in state

---

ANSWER TO QUESTION 3.

(a) Suppose L is accepted by some NFSA $M = (Q, \Sigma, \delta, s, F)$. Using M, we construct a NFSA $\hat{M} = (\hat{Q}, \Sigma, \hat{\delta}, \hat{s}, \hat{F})$, where $\hat{F}$ has only one state, so that $\hat{M}$ also accepts L. Intuitively, the idea is to add to M a new state f which will be the unique accepting state of $\hat{M}$, and add ϵ transitions from each accepting state of M to f. This construction is illustrated diagrammatically below.



M

More formally, the components of $\hat{M}$ are defined as follows: Let f be a state that is not in Q. $\hat{Q} = Q \cup \{f\}$, $\hat{s} = s$ and $\hat{F} = \{f\}$. The transition function $\hat{\delta}$ is defined as follows. For any $q \in \hat{Q}$, $a \in \Sigma \cup \{\epsilon\}$:

$$\hat{\delta}(q, a) = \begin{cases} \delta(q, a), & \text{if } q \in Q \text{ and } a \in \Sigma, \text{ or } q \in Q \Leftrightarrow F \text{ and } a = \epsilon \\ \delta(q, a) \cup \{f\}, & \text{if } q \in F \text{ and } a = \epsilon \end{cases}$$

Now we prove that $\mathcal{L}(M) = \mathcal{L}(\hat{M})$. A straightforward induction on x shows that:

Claim. For any $q, q' \in Q$ and any $x \in \Sigma^*$, $q' \in \delta^*(q, x)$ if and only if $q' \in \hat{\delta}^*(q, x)$.

Let $x \in \mathcal{L}(M)$. Thus, there is some $q \in F$ such that $q \in \delta^*(s, x)$. By the Claim, $q \in \hat{\delta}^*(s, x)$. Since $q \in F$, $f \in \hat{\delta}(q, \epsilon)$. Therefore, $f \in \hat{\delta}^*(s, x \cdot \epsilon) = \hat{\delta}^*(s, x)$. Since f is (the unique state) in $\hat{F}$, $x \in \mathcal{L}(\hat{M})$. This proves that $\mathcal{L}(M) \subseteq \mathcal{L}(\hat{M})$.

Conversely, let $x \in \mathcal{L}(\hat{M})$. Thus, $f \in \hat{\delta}^*(s, x)$. By definition of $\hat{M}$, we can only reach the state f from states in F and only by ϵ transitions from those states. Thus, there is some $q \in F$, such that $q \in \hat{\delta}^*(s, x)$. By the Claim, $q \in \delta^*(s, x)$. Since $q \in F$, $x \in \mathcal{L}(M)$. This proves that $\mathcal{L}(\hat{M}) \subseteq \mathcal{L}(M)$.

Since $\mathcal{L}(M) \subseteq \mathcal{L}(\hat{M})$ and $\mathcal{L}(\hat{M}) \subseteq \mathcal{L}(M)$, we have $\mathcal{L}(\hat{M}) = \mathcal{L}(M)$. We have therefore shown that any regular language L is accepted by a NFSA with a single accepting state.

(b) A similar claim cannot be made for *deterministic* FSA. To see this, let $\Sigma = \{0, 1\}$ and consider the language $L = \{0, 01\}$ over Σ. It is easy to see that this can be accepted by a DFSA. Let $M = (Q, \Sigma, \delta, s, F)$ be any DFSA that accepts L. We claim that M cannot have only one accepting state.

Suppose, for contradiction, that it does. Let $q = \delta(s, 0)$ and $q' = \delta^*(s, 01)$. Note that $q' = \delta(\delta(s, 0), 1) = \delta(q, 1)$. Since $0, 01 \in L$, it must be that $q, q' \in F$. Since F contains only one state, $q = q'$. Thus,

$$\delta^*(s, 011) = \delta(\delta^*(s, 01), 1) = \delta(q', 1) = \delta(q, 1) = q' \in F$$

This is a contradiction, since $011 \notin L$.

Thus the set of accepting states of any DFSA that accepts L cannot contain only one state.

ANSWER TO QUESTION 4. Let $M = (Q, \Sigma, \delta, s, F)$ be a DFSA to accept L. Recall that a state q' *is reachable from* state q if there is a string $x \in \Sigma^*$ such that $\delta^*(q, x) = q'$. Informally, this means that in the DFSA diagram there is a path from q to q'.

(a) Let F_1 be the states in Q from which some accepting state of M is reachable. More precisely, $F_1 = \{q \in Q : \text{for some } x \in \Sigma^*,\ \delta^*(q, y) \in F\}$. We claim that the DFSA $M_1 = (Q, \Sigma, \delta, s, F_1)$ accepts Prefix(L). This can be proved as follows:

$$\begin{aligned}
x \in \mathcal{L}(M_1) &\Leftrightarrow \delta^*(s, x) \in F_1 \\
&\Leftrightarrow \text{for some } y \in \Sigma^*,\ \delta^*(\delta^*(s, x), y) \in F \\
&\Leftrightarrow \text{for some } y \in \Sigma^*,\ \delta^*(s, xy) \in F \\
&\Leftrightarrow \text{for some } y \in \Sigma^*,\ xy \in L \\
&\Leftrightarrow x \in \text{Prefix}(L)
\end{aligned}$$

(b) Let F_3 be the set of accepting states of M from which we can***not*** reach any accepting state of M by following one or more transitions. More precisely, $F_3 = \{q \in F : \text{for all } y \in \Sigma^*, \text{ if } y \neq \epsilon \text{ then } \delta^*(q, y) \notin F\}$. We claim that the DFSA $M_3 = (Q, \Sigma, \delta, s, F_3)$ accepts Max(L). This can be proved as follows:

$$\begin{aligned}
x \in \mathcal{L}(M_3) &\Leftrightarrow \delta^*(s, x) \in F_3 \\
&\Leftrightarrow \delta^*(s, x) \in F, \text{ and for all } y \in \Sigma^*, \text{ if } y \neq \epsilon \text{ then } \delta^*(\delta^*(s, x), y) \notin F \\
&\Leftrightarrow \delta^*(s, x) \in F, \text{ and for all } y \in \Sigma^*, \text{ if } y \neq \epsilon \text{ then } \delta^*(s, xy) \notin F \\
&\Leftrightarrow x \in L, \text{ and for all } y \in \Sigma^*, \text{ if } y \neq \epsilon \text{ then } xy \notin L \\
&\Leftrightarrow x \in \text{Max}(L)
\end{aligned}$$
