Computer Science 238 (L5101) Fall 1999
St. George Campus University of Toronto

Answers to Homework Assignment #4

ANSWER TO QUESTION 1.

(a) This sentence says that there is some number x that is less than or equal to every number. This is true in $\mathcal{N}$ because we can choose x to be 0 which is, indeed, less than or equal to any natural number. The sentence is false in $\mathcal{Z}$ because no matter how we choose x, $x-1$ is an example of a number that is not less than or equal to x. (Note that this argument does not apply in the case of $\mathcal{N}$, because $x-1$ is *not* an element of structure's domain when $x=0$.)

(b) This sentence says that there is some number x so that every number is less than or equal to x. This is false in both $\mathcal{N}$ and $\mathcal{Z}$: No matter how we choose x, $x+1$ is an example of a number that is not less than or equal to x. (Note that if x is in the domain of $\mathcal{N}$ or $\mathcal{Z}$, then so is $x+1$.)

(c) This sentence says that for every number x there is a y that is larger. This is true in both $\mathcal{N}$ and $\mathcal{Z}$: For every x, $x+1$ is an example of a number that is greater than x.

(d) This sentence says that for every number x there is a y that is smaller. This is false in $\mathcal{N}$ because if we choose $x=0$, no natural number is smaller than it. On the other hand, the sentence is true in $\mathcal{Z}$ because for any x, $x-1$ is an example of a number that is less than x.

(d) This sentence says that for any numbers x and y, if $x<y$ then $x^2<y^2$. This is true in $\mathcal{N}$. It can be proved in a variety of ways (including induction on y). Here is a simple proof: $x^2-y^2=(x-y)(x+y)$. $x-y$ is negative (since $x<y$) and $x+y$ is positive (since x,y are both natural numbers), so their product is negative. Thus, $x^2-y^2<0$, i.e., $x^2<y^2$.

On the other hand, this sentence is false in $\mathcal{Z}$: If we take $x=-2$ and $y=0$, we have $x<y$ but $x^2>y^2$.

ANSWER TO QUESTION 2.

(a) Goldbach's conjecture (which should read that every *even* integer greater than 2 is the sum of two primes): $\forall x\,\forall u\,\Big(\big(S(u,u,x)\wedge L(\mathbf{2},x)\big)\rightarrow \exists y\,\exists z\,\big(Prime(y)\wedge Prime(z)\wedge S(y,z,x)\big)\Big)$

(b) Twin prime conjecture: $\forall x\,\exists y\,\Big(L(x,y)\wedge Prime(y)\wedge \exists z\,\big(S(y,\mathbf{2},z)\wedge Prime(z)\big)\Big)$

(c) Division theorem:

$$\forall x\,\forall y\,\Bigg(\neg{\approx}(y,0)\;\rightarrow \exists q\,\exists r\,\exists u\,\Bigg(\big(P(q,y,u)\wedge S(u,r,x)\wedge L(r,y)\big)\wedge$$
$$\forall q'\,\forall r'\,\forall u'\,\Big(\big(P(q',y',u')\wedge S(u',r',x)\wedge L(r',y)\big)\rightarrow\big({\approx}(q,q')\wedge{\approx}(r,r')\big)\Big)\Bigg)\Bigg)$$

Roughly speaking, the part of the first line after the implication symbol asserts the existence of a quotient q and a remainder r such that $x=q\cdot y+r$ and $r<y$ (note that we don't have to assert that $0\le r$ since our domain is the natural numbers), and the second line asserts that the quotient and remainder are unique — in the sense that any two numbers q' and r' with the same property must be equal, respectively, to q and r.

ANSWER TO QUESTION 3.

(a) is false. For example, suppose $A(x)$ is the predicate "x is an odd number that is a multiple of 2", and $B(x)$ is the predicate "x is even". Then $\forall x\, \big(A(x) \to B(x)\big)$ is true (because, for each x $A(x)$ is false and so the conditional $A(x) \to B(x)$ is true), but obviously $\exists x\, \big(A(x) \wedge B(x)\big)$ is false, because no number x can be both odd and even.

(b) is true.

$$
\begin{array}{lll}
 & \exists x\, \big(A(x) \to \neg B(x)\big) & \\
\text{LEQV} & \neg\forall x\, \neg\big(A(x) \to \neg B(x)\big) & \text{[by duality of quantifiers]} \\
\text{LEQV} & \neg\forall x\, \neg\big(\neg A(x) \vee \neg B(x)\big) & \text{[by } \to\text{-rule]} \\
\text{LEQV} & \neg\forall x\, \big(A(x) \wedge B(x)\big) & \text{[by DeMorgan and double negation]}
\end{array}
$$

(c) is false. For example, let $A(x)$ be the predicate "x is male". Then $\exists x\, A(x) \wedge \exists \neg A(x)$ is true since it asserts that there is someone who is male and someone who is not. However, $\exists x\, \big(A(x) \wedge \neg A(x)\big)$ is false, since it asserts that there is someone who is both male and not male.

(d) is true. $\exists x\, A(x) \vee \exists x\, \neg A(x)$ asserts that some x satisfies $A(x)$ or some x satisfies $\neg A(x)$. This is true no matter what predicate the predicate symbol A stands for — i.e., it is a valid formula.

The formula $\exists x\, \big(A(x) \vee \neg A(x)\big)$ asserts that some x satisfies either $A(x)$ or $\neg A(x)$. This is also true no matter what predicate the symbol A stands for. Since both formulas are valid, they are logically equivalent.

(e) is false. For example, let P be the formula $A(x)$ and Q be the formula $\forall y A(y)$. In this case, the formula $\forall x\, P \leftrightarrow Q$ is

$$F: \qquad \forall x\, A(x) \leftrightarrow \forall y\, A(y).$$

This formula is valid by the renaming logical equivalence and Theorem 5.11(b). On the other hand, the formula $\forall x\, (P \leftrightarrow Q)$ is

$$G: \qquad \forall x\, \big(A(x) \leftrightarrow \forall y\, A(y)\big).$$

This formula is not valid: For example, if $A(x)$ stands for the predicate "$x = 0$", G is false. Since F is valid and G is not, the two formulas cannot be logically equivalent.

ANSWER TO QUESTION 4.

(a) $\exists s\, \big(\mathit{Student}(s, n, a) \wedge \exists m\; \mathit{Takes}(s, \text{CSCB38}, 1999, m)\big)$

(b) $\exists c\, (\mathit{Course}(c, n) \wedge \exists y\; \mathit{Teaches}(\text{MacLean}, c, y))$

(c) $\exists s\, \exists a\, \exists y\, \big(\mathit{Student}(s, n, a) \wedge \mathit{Takes}(s, \text{CSCB70}, y, \text{A}) \wedge \mathit{Teaches}(\text{Panario}, \text{CSCB70}, y)\big)$
(The interpretation here is that we want the names of all students who received an A in CSCB70 is *some* year when Panario taught the course.)

(d) $\exists c\, \Big(\mathit{Course}(c, n) \wedge \forall p\, \forall y\, (\mathit{Teaches}(p, c, y) \to \approx(p, \text{Rackoff}))\Big)$

(e) $\exists c\, \exists p\, \Big(\mathit{Course}(c, n) \wedge \mathit{Teaches}(p, c, 1999) \wedge \forall c'\, \forall y\, \big(\mathit{Teaches}(p, c', y) \to \neg\exists s\; \mathit{Takes}(s, c', y, \text{F})\big)\Big)$

ANSWER TO QUESTION 5. The Roman numerals refer to rules on pages 154–157 of the course notes.

	$\exists x \, \neg \exists y \, A(x,y) \;\to\; \exists y \, \big(B(x,y) \leftrightarrow \forall z \, C(z)\big)$	
LEQV	$\neg \forall x \, \exists y \, A(x,y) \;\to\; \exists y \, \big(B(x,y) \leftrightarrow \forall z \, C(z)\big)$	[Ia]
LEQV	$\forall x \, \exists y \, A(x,y) \;\vee\; \exists y \, \big(B(x,y) \leftrightarrow \forall z \, C(z)\big)$	[$\to$ rule]
LEQV	$\forall u \, \exists y \, A(u,y) \;\vee\; \exists y \, \big(B(x,y) \leftrightarrow \forall z \, C(z)\big)$	[III]
LEQV	$\forall u \, \exists y \, \Big(A(u,y) \;\vee\; \exists y \, \big(B(x,y) \leftrightarrow \forall z \, C(z)\big)\Big)$	[commutativity of $\vee$ and IIb]
LEQV	$\forall u \, \exists y \, \Bigg(A(u,y) \;\vee\; \exists y \, \Big(\big(B(x,y) \wedge \forall z \, C(z)\big) \vee \big(\neg B(x,y) \wedge \neg \forall z \, C(z)\big)\Big)\Bigg)$	[$\leftrightarrow$ rule]
LEQV	$\forall u \, \exists y \, \Bigg(A(u,y) \;\vee\; \exists y \, \Big(\forall z \, \big(B(x,y) \wedge C(z)\big) \vee \exists z \, \big(\neg B(x,y) \wedge \neg C(z)\big)\Big)\Bigg)$	[various rules]
LEQV	$\forall u \, \exists y \, \Bigg(A(u,y) \;\vee\; \exists y \, \forall z \, \Big(\big(B(x,y) \wedge C(z)\big) \vee \exists z \, \big(\neg B(x,y) \wedge \neg C(z)\big)\Big)\Bigg)$	[commutativity of $\vee$ and IIb]
LEQV	$\forall u \, \exists y \, \Bigg(A(u,y) \;\vee\; \exists v \, \forall z \, \Big(\big(B(x,v) \wedge C(z)\big) \vee \exists z \, \big(\neg B(x,v) \wedge \neg C(z)\big)\Big)\Bigg)$	[III]
LEQV	$\forall u \, \exists y \, \exists v \, \forall z \, \Bigg(A(u,y) \;\vee\; \Big(\big(B(x,v) \wedge C(z)\big) \vee \exists z \, \big(\neg B(x,v) \wedge \neg C(z)\big)\Big)\Bigg)$	[IIb, applied twice]
LEQV	$\forall u \, \exists y \, \exists v \, \forall z \, \Bigg(A(u,y) \;\vee\; \Big(\big(B(x,v) \wedge C(z)\big) \vee \exists w \, \big(\neg B(x,v) \wedge \neg C(w)\big)\Big)\Bigg)$	[III]
LEQV	$\forall u \, \exists y \, \exists v \, \forall z \, \Bigg(A(u,y) \;\vee\; \exists w \, \Big(\big(B(x,v) \wedge C(z)\big) \vee \big(\neg B(x,v) \wedge \neg C(w)\big)\Big)\Bigg)$	[IIb]
LEQV	$\forall u \, \exists y \, \exists v \, \forall z \, \exists w \, \Bigg(A(u,y) \;\vee\; \Big(\big(B(x,v) \wedge C(z)\big) \vee \big(\neg B(x,v) \wedge \neg C(w)\big)\Big)\Bigg)$	[IIb]
LEQV	$\forall u \, \exists y \, \exists v \, \forall z \, \exists w \, \Bigg(A(u,y) \;\vee\; \Big(\neg\big(\neg B(x,v) \vee \neg C(z)\big) \vee \neg\big(B(x,v) \vee C(w)\big)\Big)\Bigg)$	[DeMorgan]
