Computer Science CSC238 L5101 — October 27, 1999

St. George Campus Campus — University of Toronto

Homework Assignment #3

Due: Tuesday, November 9, 1999, by 6:10 pm (beginning of tutorial)

On the cover page of your assignment, you must write **and sign** *the following statement: "I have read and understood the policy on collaboration in homework stated in the Course Information handout." Without such a signed statement your homework will not be marked.*

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1. (10 marks) Let $S(n)$ denote the time to compute the product of two $n \times n$ matrices using Strassen's algorithm (see below). Then, we can write

$$S(n) = \begin{cases} C, & \text{if } n = 1 \\ 7S\left(\lceil \frac{n}{2} \rceil\right) + D\left(\lceil \frac{n}{2} \rceil\right)^2, & \text{if } n > 1 \end{cases}$$

where C and D are constants.

(a) Find an exact solution (i.e., closed-form formula) for the above recurrence when $n = 2^k$, for some $k \in \mathbb{N}$, and prove that your solution is correct. Note that we require an exact solution here, so the general theorem for solving divide-and-conquer recurrences (as described in Section 3.3) is not applicable.

(b) Consider now any $n \in \mathbb{N}$. Give a Θ-expression for $S(n)$. (**Hint:** Use the result from part (a) and the methodology developed in Section 3.3, specifically Theorem 3.8.)

Note: Although you do not need the algorithm to do this exercise, Strassen's algorithm is shown below. You may find it instructive to derive the recurrence $S(n)$ above.

```
STRASSEN(P, Q: [n × n] matrix)
    if n = 1 then
        R := P × Q        { integer multiplication }
    else begin
        parity := n mod 2;
        if parity = 1 then
            add a row and a column of 0's to P and Q;
            n := n + 1
        end if;
        let P := [ P11  P12 ]  and Q := [ Q11  Q12 ];
                 [ P21  P22 ]            [ Q21  Q22 ]
        M1 := STRASSEN(P12 − P22, Q21 + Q22);
        M2 := STRASSEN(P11 + P22, Q11 + Q22);
        M3 := STRASSEN(P11 − P21, Q11 + Q12);
        M4 := STRASSEN(P11 + P12, Q22);
        M5 := STRASSEN(P11, Q12 − Q22);
        M6 := STRASSEN(P22, Q21 − Q11);
        M7 := STRASSEN(P21 + P22, Q11);
        R11 := M1 + M2 − M4 + M6;
        R12 := M4 + M5;
        R21 := M6 + M7;
        R22 := M2 − M3 + M5 − M7;
        let R := [ R11  R12 ];
                 [ R21  R22 ]
        if parity = 1 then
            remove the last row and the last column from R;
            n := n − 1
        end if
    end;
    return R
END
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2. (10 marks) Let x and y be three-bit natural numbers. We can represent the value of x by using three propositional variables: x_2 (to denote the value of the most significant bit of x), x_1 (to represent the middle bit of x), and x_0 (to denote the value of the least significant bit of x). More precisely, a truth assignment τ to x_0, x_1 and x_2 represents the (three-bit) number $\tau(x_0) + 2\tau(x_1) + 4\tau(x_2)$. Similarly, we can represent the value of y by using three propositional variables y_2, y_1, and y_0.

The sum $x + y$ is a four-bit natural number. For each of the four bits in the sum $x + y$, write a propositional formula (with propositional variables $x_0, x_1, x_2, y_0, y_1, y_2$) which represents the value of that bit. You should write four formulas F_0, F_1, F_2, F_3 where F_0 represents the low-order bit of the sum and F_3 represents the high-order bit. Explain how you obtained your propositional formulas. Your formulas may contain any of the propositional connectives we have discussed ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus, |, \downarrow$).

Example: Consider the truth assignment which gives x the value 6 and y the value 5 That is, the variables $x_2, x_1, x_0, y_2, y_1, y_0$ get, respectively, the values $1, 1, 0, 1, 0, 1$. Since the sum $6 + 5 = 11$, the values of the formulas F_3, F_2, F_1, F_0 under this truth assignment should be (respectively) $1, 0, 1, 1$.

Hint: This can certainly be done with a truth table that involves six propositional variables (and therefore has 64 rows). This is too much boring work. Instead, think about how to add numbers in binary, and represent the algorithm in terms of logical operations. This will naturally lead you to the four formulas, via a probably faster (and surely intellectually more interesting) path.

3. (10 marks)

(a) Using only logical equivalences from Section 4.6 (and, in particular, using no truth tables), prove that $(x \leftrightarrow \neg y) \rightarrow \neg(x \rightarrow y)$ is logically equivalent to $y \rightarrow x$.

(b) Is $(x \rightarrow y) \wedge (x \rightarrow z)$ logically equivalent to $x \rightarrow (y \wedge z)$? Justify your answer.

(c) Is $(y \rightarrow x) \wedge (z \rightarrow x)$ logically equivalent to $(y \wedge z) \rightarrow x$? Justify your answer.

4. (10 marks) Let $P_1, P_2, \ldots, P_n$ be arbitrary propositional formulas. Prove that, for any $n \geq 2$, the formula $P_1 \rightarrow (P_2 \rightarrow (\cdots \rightarrow (P_{n-1} \rightarrow P_n) \cdots))$ is logically equivalent to $(P_1 \wedge P_2 \wedge \cdots \wedge P_{n-1}) \rightarrow P_n$. (**Hint:** use induction on n.)

5. (10 marks) Following is the truth table for Parity(x, y, z),

x	y	z	Parity(x, y, z)
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

(a) Construct CNF and DNF formulas which represent this function.

(b) Construct a formula that represents this function and uses only NOR gates ($\downarrow$).