

Answers to Homework Assignment #2

**ANSWER TO QUESTION 1.** Let  $a, b, x$  be such that the Precondition holds. So  $a, b \in \mathbb{N}$ ,  $b > 0$ ,  $x > a$ , and for some  $k \in \mathbb{N}$ ,  $x = b \cdot 2^k$  and  $y = 2^k$ . We first prove the following Loop Invariant lemma.

*Lemma:* After every iteration of the loop:

$a = qb + r$  and

$r \geq 0$  and

if  $r \geq b$  then

$[2x > r$  and for some  $k \in \mathbb{N}$ ,  $x = b \cdot 2^k$  and  $y = 2^k]$ .

*Proof:* As usual, let  $q_n, r_n, x_n$  be the values of  $q, r, x$  after  $n$  iterations of the loop.

Let  $P(n)$  be: If the loop is iterated at least  $n$  times, then

$a = q_nb + r_n$  and

$r_n \geq 0$  and

if  $r_n \geq b$  then

$[2x_n > r_n$  and for some  $k \in \mathbb{N}$ ,  $x_n = b \cdot 2^k$  and  $y_n = 2^k]$ .

We will prove that  $P(n)$  holds for all  $n \in \mathbb{N}$ , by (simple) induction.

**BASIS:** We wish to show  $P(0)$ .

Since  $q_0 = 0$  and  $r_0 = a$ , we have  $a = q_0b + r_0$ .

By the Precondition,  $a \geq 0$ , so  $r_0 \geq 0$ .

Whether or not  $r_0 \geq b$ , we see the following.

By the Precondition,  $x_0 > a \geq 0$ , so  $2x_0 > a = r_0$ .

Also, by the Precondition, for some  $k \in \mathbb{N}$ ,  $x_0 = b \cdot 2^k$  and  $y_0 = 2^k$ .

**INDUCTION STEP:** Let  $i$  be an arbitrary natural number. Suppose that  $P(i)$  holds. We will show  $P(i+1)$ .

Assume the loop is iterated at least  $i+1$  times, and hence  $r_i \geq b$ .

Then by  $P(i)$ ,  $2x_i > r_i$ ; also, we can let  $k \in \mathbb{N}$  be such that  $x_i = b \cdot 2^k$  and  $y_i = 2^k$ .

We have two cases to consider.

**CASE 1.**  $r_i \geq x_i$ .

Then  $r_{i+1} = r_i - x_i \geq 0$  and  $q_{i+1} = q_i + y_i$ .

By the induction hypothesis  $P(i)$ ,  $a = q_ib + r_i$ .

So  $a = q_ib + r_i = q_ib + (y_ib - x_i) + r_i = (q_i + y_i)b + (r_i - x_i) = q_{i+1}b + r_{i+1}$ .

Now assume that  $r_{i+1} \geq b$ . Since  $2x_i > r_i$ , we have

$x_i > r_i - x_i = r_{i+1} \geq b$ ,

so  $x_i > b$ , so  $k \geq 1$ .

So  $x_{i+1} = x_i \text{ div } 2 = x_i/2 = b \cdot 2^{k-1}$ , and  $y_{i+1} = y_i \text{ div } 2 = y_i/2 = 2^{k-1}$ .

Also,  $2x_{i+1} = x_i > r_{i+1}$ .

**CASE 2.**  $x_i > r_i$ .

Then  $r_{i+1} = r_i$  and  $q_{i+1} = q_i$ .

Since  $P(i)$  tells us that  $a_i = q_ib + r_i$  and  $r_i \geq 0$ , we have

$a = q_{i+1}b + r_{i+1}$  and  $r_{i+1} \geq 0$ .

We have  $x_i > r_i \geq b$ , so  $x_i > b$ , so  $k \geq 1$ .

So  $x_{i+1} = x_i \text{ div } 2 = x_i/2 = b \cdot 2^{k-1}$ , and  $y_{i+1} = y_i \text{ div } 2 = y_i/2 = 2^{k-1}$ .

Also,  $2x_{i+1} = x_i > r_i = r_{i+1}$ .

This concludes the proof of the Lemma. Partial correctness can now be trivially proved as follows. If and when the program halts, the Lemma tells us that  $a = qb + r$  and  $r \geq 0$ . Because the program halted, we have  $r < b$ . By the definition of “div” on Page 28, we have  $q = a \text{ div } b$ .

To prove termination, we will do a proof by contradiction. Assume that the program does not terminate. Since the program doesn't terminate, we see from the exit condition that  $r_i \geq b$  for every  $i$ . The Loop Invariant therefore tells us that  $x_i \geq b > 0$  for every  $i$ . Since  $x_i > 0$  and  $x_{i+1} = x_i \text{ div } 2$ , we have  $x_{i+1} < x_i$  for every  $i$ . Since the quantity  $x$  is decreasing and bounded below (by 0), the termination principle tells us the program halts, contradicting our assumption that it didn't. So it does.

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## ANSWER TO QUESTION 2.

(a) Let  $A$  be an integer array; we take it for granted that the program never changes any of the elements of  $A$ . Correctness of the program follows from the induction proof below.

Let  $P(n)$  be: For all integers  $F$  and  $L$  such that  $L - F = n$ , if  $\text{MIN}(A, F, L)$  is invoked, then a (integer) value  $u$  is returned such that

$F \leq u \leq L$ , and

$A(u)$  is the smallest value in  $\{A(v) \mid F \leq v \leq L\}$ , and

$A(u) < A(v)$  for every  $v$  such that  $u < v \leq L$ .

We will prove by (complete) induction that  $P(n)$  holds for all  $n \in \mathbb{N}$ . So let  $i \in \mathbb{N}$  be such that for all integers  $j$ , if  $0 \leq j < i$ , then  $P(j)$  holds. We wish to show that  $P(i)$  holds.

CASE 1.  $i = 0$ . (This is, in effect, the base case.)

Let  $F$  and  $L$  be integers such that  $L - F = i$ . So  $F = L$ . The program halts immediately, returning the value  $F$ , and clearly

$F \leq F \leq F$ , and

$A(F)$  is the smallest value in  $\{A(v) \mid F \leq v \leq F\}$ , and

$A(F) < A(v)$  for every  $v$  such that  $F < v \leq F$  (since  $\{v \mid F < v \leq F\} = \emptyset$ ).

CASE 2.  $i > 0$ . (These are, in effect, the induction cases.)

Let  $F$  and  $L$  be integers such that  $L - F = i$ . So  $F < L$ .

Let  $M = (F + L) \text{ div } 2$ . By the Lemma on Page 65,  $F \leq M < L$ .

So we have  $0 \leq M - F < i$  and  $0 \leq L - (M + 1) < i$ . So the induction hypothesis tells us that  $\text{MIN}(A, F, M)$  returns a value  $x$  and

$\text{MIN}(A, M + 1, L)$  returns a value  $y$  such that

$F \leq x \leq M$ , and

$A(x)$  is the smallest value in  $\{A(v) \mid F \leq v \leq M\}$ , and

$A(x) < A(v)$  for every  $v$  such that  $x < v \leq M$ .

and

$M + 1 \leq y \leq L$ , and

$A(y)$  is the smallest value in  $\{A(v) \mid M + 1 \leq v \leq L\}$ , and

$A(y) < A(v)$  for every  $v$  such that  $y < v \leq L$ .

We have two cases to consider.

SUBCASE 1.  $A(y) \leq A(x)$ . In this case, the program halts and returns the value  $y$ .

Since  $F < M + 1 \leq y \leq L$ , we have  $F \leq y \leq L$ .

Since  $A(x)$  is the smallest value in  $\{A(v) \mid F \leq v \leq M\}$ , and  $A(y) \leq A(x)$ , and

$A(y)$  is the smallest value in  $\{A(v) \mid M + 1 \leq v \leq L\}$ , we have

$A(y)$  is the smallest value in  $\{A(v) \mid F \leq v \leq L\}$ .

Lastly, we have already stated that  $A(y) < A(v)$  for every  $v$  such that  $y < v \leq L$ .

SUBCASE 2.  $A(x) < A(y)$ . In this case, the program halts and returns the value  $x$ . Since  $F \leq x \leq M < L$ , we have  $F \leq x \leq L$ .

Since  $A(x)$  is the smallest value in  $\{A(v) | F \leq v \leq M\}$ , and  $A(x) \leq A(y)$ , and  $A(y)$  is the smallest value in  $\{A(v) | M + 1 \leq v \leq L\}$ , we have

$A(x)$  is the smallest value in  $\{A(v) | F \leq v \leq L\}$ .

Lastly, since  $A(x) < A(y)$ , we have that

$A(x)$  is (strictly) smaller than every element of  $\{A(v) | M + 1 \leq v \leq L\}$ ,

as well as being smaller than every member of  $\{A(v) | x < v \leq M\}$ .

Therefore,  $A(x) < A(v)$  for every  $v$  such that  $x < v \leq L$ .

(b) For some positive constants  $c_1$  and  $c_2$ , we have the recurrence

$$\begin{aligned} T(1) &\leq c_1 \\ T(n) &\leq T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + c_2 \text{ for every } n \in \mathbb{N}, n > 1. \end{aligned}$$

(c) The general theorem gives us  $T(n) \in O(n)$ .

### ANSWER TO QUESTION 3.

(a) We will let  $c = 8$  and  $n_0 = 1$ , and prove that for all integers  $n \geq 1$ ,  $T(n) \leq 8n^{\log_3 4}$ . We will actually prove the stronger statement that for all integers  $n \geq 1$ ,  $T(n) \leq 8n^{\log_3 4} - 3n - 4$ ; this gives us what we want since  $8n^{\log_3 4} - 3n - 4 \leq 8n^{\log_3 4}$  whenever  $n$  is nonnegative.

Let  $P(n)$  be:  $T(n) \leq 8n^{\log_3 4} - 3n - 4$ .

We will prove by (complete) induction that  $P(n)$  holds for all integers  $n \geq 1$ .

So let  $i$  be an integer  $\geq 1$  such that for all integers  $j$ , if  $1 \leq j < i$ , then  $P(j)$ . We will prove  $P(i)$ .

CASE 1.  $1 \leq i \leq 2$ . (These are the base cases.)

$P(1)$  holds since  $1 = T(1) = 8 \cdot 1^{\log_3 4} - 3 \cdot 1 - 4$ .

To show  $P(2)$ , we calculate  $T(2) = 4T(\lfloor 2/3 \rfloor) + 2 + 3 = 4T(0) + 5 = 5$ , and  $8 \cdot 2^{\log_3 4} - 3 \cdot 2 - 4 \geq 8 \cdot 2^1 - 10 = 6 \geq 5$ . So  $P(2)$  holds.

CASE 2.  $i \geq 3$ . (These are the induction cases.)

Since  $i \geq 3$ , we have  $1 \leq \lfloor i/3 \rfloor < i$ . (EXERCISE!)

So  $P(\lfloor i/3 \rfloor)$  holds. Using the fact that  $\lfloor \frac{i}{3} \rfloor \leq \frac{i}{3}$  and  $\lfloor \frac{i}{3} \rfloor \geq \frac{i-2}{3}$ , we have (since  $i > 1$ )

$$T(i) = 4T(\lfloor i/3 \rfloor) + i + 3 \leq 4(8\lfloor i/3 \rfloor^{\log_3 4} - 3\lfloor i/3 \rfloor - 4) + i + 3 \leq 4(8(i/3)^{\log_3 4} - 3((i-2)/3) - 4) + i + 3 = 8i^{\log_3 4} - 3i - 5 \leq 8i^{\log_3 4} - 3i - 4.$$

(b) We will prove that if  $n = 3^k$  for  $k \in \mathbb{N}$ , then  $T(n) \leq 5n^{\log_3 4} - 3n - 1$ . This gives us what we want since  $5n^{\log_3 4} - 3n - 1 \leq 5n^{\log_3 4}$  whenever  $n$  is nonnegative. (Although we could use simple induction to prove this as in Part (c) below, we choose to use complete induction.)

Let  $P(n)$  be: If  $n = 3^k$  for  $k \in \mathbb{N}$ , then  $T(n) \leq 5n^{\log_3 4} - 3n - 1$ . We will prove that  $P(n)$  holds for every integer  $n \geq 1$ , by complete induction.

So let  $i \geq 1$  be an integer such that for all integers  $j$ , if  $1 \leq j < i$ , then  $P(j)$  holds. We wish to show that  $P(i)$  holds. Assume that  $i = 3^k$  for  $k \in \mathbb{N}$ .

CASE 1.  $k = 0$ ,  $i = 1$ . (This is, in effect, the base case.)

$T(i) = 1 = 5 \cdot 1^{\log_3 4} - 3 \cdot 1 - 1$ , so  $P(i)$  holds.

CASE 2.  $k \geq 1$ ,  $i = 3^k$ . (These are, in effect, the induction cases.)

Note that  $1 \leq \lfloor i/3 \rfloor < i$ , so  $P(\lfloor i/3 \rfloor)$  holds. Since  $\lfloor i/3 \rfloor$  is a power of 3 and  $\lfloor i/3 \rfloor = i/3$ , we have

$$T(\lfloor i/3 \rfloor) \leq 5(i/3)^{\log_3 4} - 3(i/3) - 1 = (5/4)i^{\log_3 4} - i - 1.$$

Since  $i > 1$ ,

$$T(i) = 4T(\lfloor i/3 \rfloor) + i + 3 \leq 4((5/4)i^{\log_3 4} - i - 1) + i + 3 = 5i^{\log_3 4} - 3i - 1,$$

so  $P(i)$  holds.

(c) Let  $c = \frac{12}{2^{\log_3 4}}$ . We will prove the following Lemma.

*Lemma:* If  $n = 2 \cdot 3^k$  for  $k \in \mathbb{N}$ , then  $T(n) \geq cn^{\log_3 4} - 3n - 1$ .

Before proving the Lemma, let us see how to use it to get the result we want. Let  $c' < c$ ; we want to show that for infinitely many  $n \in \mathbb{N}$ ,  $T(n) > c'n^{\log_3 4}$ . Given the Lemma, it is sufficient to exhibit some  $n_0$  such that for all  $n > n_0$ ,  $cn^{\log_3 4} - 3n - 1 > c'n^{\log_3 4}$ . Assuming we choose  $n_0 \geq 1$  it is sufficient to show that for all  $n > n_0$ ,  $cn^{\log_3 4} - 4n \geq c' \cdot n^{\log_3 4}$ ; this will hold (since  $c - c' > 0$ ) if  $n^{\log_3 4} \geq 4n/(c - c')$ , which (since  $\log_3 4 > 1.2$ ) holds if  $n \geq (4/(c - c'))^5$ .

So we can let  $n_0$  be the maximum of 1 and  $(4/(c - c'))^5$ .

*Proof of Lemma:* (Although we could use complete induction to prove this as in Part (b) above, we choose to use simple induction.)

Let  $P(k)$  be: For  $n = 2 \cdot 3^k$ ,  $T(n) \geq cn^{\log_3 4} - 3n - 1$ . We will prove that  $P(k)$  holds for every integer  $k \geq 0$ , by simple induction.

**BASIS:** We wish to show  $P(0)$ . Let  $k = 0$  and  $n = 2 \cdot 3^k = 2$ .

$$T(2) = 4T(\lfloor 2/3 \rfloor) + 2 + 3 = 4T(0) + 5 = 5. \text{ Also,}$$

$$cn^{\log_3 4} - 3n - 1 = \frac{12}{2^{\log_3 4}} \cdot 2^{\log_3 4} - 7 = 5 = T(2).$$

So  $P(0)$  holds.

**INDUCTION STEP:** Let  $i$  be an arbitrary natural number. Suppose that  $P(i)$  holds. We will show  $P(i+1)$ .

We are given that

$$T(2 \cdot 3^i) \geq c(2 \cdot 3^i)^{\log_3 4} - 3(2 \cdot 3^i) - 1. \text{ Since } 2 \cdot 3^{i+1} > 1,$$

$$T(2 \cdot 3^{i+1}) = 4T(2 \cdot 3^i) + (2 \cdot 3^{i+1}) + 3 \geq 4(c(2 \cdot 3^i)^{\log_3 4} - 3(2 \cdot 3^i) - 1) + (2 \cdot 3^{i+1}) + 3 = 3^{\log_3 4} c(2 \cdot 3^i)^{\log_3 4} - 4 \cdot 3 \cdot (2 \cdot 3^i) - 4 + (2 \cdot 3^{i+1}) + 3 = c(2 \cdot 3^{i+1})^{\log_3 4} - 3(2 \cdot 3^{i+1}) - 1.$$

So  $P(i+1)$  holds.

(d) (**Extra Credit**) Let  $c$  be the constant from the previous part.

Let  $P(n)$  be:  $T(n) \leq cn^{\log_3 4} - 3n - 1$ .

We will prove by complete induction that  $P(n)$  holds for every integer  $n \geq 1$ .

Let  $i \geq 1$  be an integer such that for every integer  $j$ , if  $1 \leq j < i$  then  $P(j)$ . We will prove  $P(i)$ .

**CASE 1.**  $1 \leq i \leq 2$ .

From the proof in Part (b) above we know that  $T(1) \leq 5 \cdot 1^{\log_3 4} - 3 \cdot 1 - 1 < c \cdot 1^{\log_3 4} - 3 \cdot 1 - 1$ .

In the Basis step of Part (c) above we proved that  $T(2) = c \cdot 2^{\log_3 4} - 3 \cdot 2 - 1$ .

**CASE 2.**  $i \geq 3$  and  $i$  is divisible by 3.

Note that  $1 \leq \lfloor i/3 \rfloor < i$ , so  $P(\lfloor i/3 \rfloor)$  holds. Since  $\lfloor i/3 \rfloor = i/3$ , we have

$$T(\lfloor i/3 \rfloor) \leq c(i/3)^{\log_3 4} - 3(i/3) - 1 = (c/4)i^{\log_3 4} - i - 1.$$

Since  $i > 1$ ,

$$T(i) = 4T(\lfloor i/3 \rfloor) + i + 3 \leq 4((c/4)i^{\log_3 4} - i - 1) + i + 3 = ci^{\log_3 4} - 3i - 1,$$

so  $P(i)$  holds.

**CASE 3.**  $i \geq 4$  and  $i$  is *not* divisible by 3.

Since  $i \geq 3$ , we have  $1 \leq \lfloor i/3 \rfloor < i$ , and so  $P(\lfloor i/3 \rfloor)$  holds.

We also have  $\lfloor i/3 \rfloor \geq (i-2)/3$ , and, since  $i$  isn't divisible by 3,  $\lfloor i/3 \rfloor \leq (i-1)/3$ .

Letting  $D_i = (i/3)^{\log_3 4} - ((i-1)/3)^{\log_3 4}$ , we have

$$T(\lfloor i/3 \rfloor) \leq c\lfloor i/3 \rfloor^{\log_3 4} - 3\lfloor i/3 \rfloor - 1 \leq c((i-1)/3)^{\log_3 4} - 3((i-2)/3) - 1 =$$

$$c(i/3)^{\log_3 4} - c((i/3)^{\log_3 4} - ((i-1)/3)^{\log_3 4}) - i + 1 = c(i/3)^{\log_3 4} - cD_i - i + 1.$$

Since  $i > 1$ , we have

$$T(i) = 4T(\lfloor i/3 \rfloor) + i + 3 \leq 4[c(i/3)^{\log_3 4} - cD_i - i + 1] + i + 3 = ci^{\log_3 4} - 4cD_i - 3i + 7.$$

We wish to show that this last quantity is  $\leq ci^{\log_3 4} - 3i - 1$ . It therefore suffices to show that  $4cD_i \geq 8$ . By taking the derivative of  $D_i$ , we see that  $D_i$  is increasing as long as  $i \geq 1$ .

Since  $i \geq 4$ , we have that  $4cD_i \geq 4cD_4$ , so it suffices to check that  $4cD_4 \geq 8$ . We can check on a calculator that this is true.

#### ANSWER TO QUESTION 4.

(a) *Proof of Conjecture:* Let  $c_1, c_2$  be positive numbers and let  $n_1, n_2 \in \mathbb{N}$  such that:

$$f_1(n) \leq c_1 g_1(n) \text{ for all } n \geq n_1, \text{ and}$$

$$f_2(n) \leq c_2 g_2(n) \text{ for all } n \geq n_2.$$

Then letting  $c_3$  be the maximum of  $c_1, c_2$  and letting  $n_3$  be the maximum of  $n_1, n_2$ , we have that for all  $n \geq n_3$ ,

$$(f_1 + f_2)(n) = f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n) \leq c_3 g_1(n) + c_3 g_2(n) = c_3(g_1(n) + g_2(n)) = c_3(g_1 + g_2)(n).$$

(b) Here is an example that shows that this conjecture is false.

Define  $f_1(n) = g_1(n) = 0$ , and  $f_2(n) = g_2(n) = 1$  for every  $n \in \mathbb{N}$ .

Clearly  $f_1 \in O(g_1)$  and  $f_2 \in O(g_2)$ .

However,  $(f_1 + f_2)(n) = 1$  and  $(g_1 \cdot g_2)(n) = 0$  for every  $n \in \mathbb{N}$ , and so it is easy to see that  $(f_1 + f_2) \notin O(g_1 \cdot g_2)$ .

(c) Here is an example that shows that this conjecture is false.

For every even  $n \in \mathbb{N}$ , define  $f_1(n) = g_2(n) = n$  and  $f_2(n) = g_1(n) = n^2$ , and

for every odd  $n \in \mathbb{N}$ , define  $f_1(n) = g_2(n) = n^2$  and  $f_2(n) = g_1(n) = n$ .

Then  $f_1 + f_2 = g_1 + g_2$  so  $f_1 + f_2 \in O(g_1 + g_2)$ .

However, for infinitely many  $n$  (namely the odd numbers)  $f_1(n) = n^2$  and  $g_1(n) = n$ , so it is easy to see that  $f_1 \notin O(g_1)$ . Similarly, we can show that  $f_2 \notin O(g_2)$ .