Computer Science 238 Fall 1999
Erindale Campus University of Toronto

Answers to Homework Assignment #1

ANSWER TO QUESTION 1. We will prove that for $c = 7$, $f(n) \leq c \cdot 5^n$ for all $n \in \mathbb{N}$. This clearly follows from the following lemma.

Lemma: For all $n \in \mathbb{N}$, $f(n) \leq 7 \cdot 5^n - 3^n$.

Proof: Let $P(n)$ be: $f(n) \leq 7 \cdot 5^n - 3^n$.
We will prove that $P(n)$ holds for all $n \in \mathbb{N}$, by (simple) induction.

BASIS: We wish to show $P(0)$, that is, that $f(0) \leq 7 \cdot 5^0 - 3^0$. This is clear, since both sides of the inequality equal 6.

INDUCTION STEP: Let i be an arbitrary natural number. Suppose that $P(i)$ holds, i.e., $f(i) \leq 7 \cdot 5^i - 3^i$. We must show that $P(i+1)$ holds, i.e., $f(i+1) \leq 7 \cdot 5^{i+1} - 3^{i+1}$. We have:

$$\begin{aligned} f(i+1) &= 5f(i) + 3^i \\ &\leq 5(7 \cdot 5^i - 3^i) + 3^i \\ &= 7 \cdot 5^{i+1} - 5 \cdot 3^i + 3^i && \text{by the induction hypothesis} \\ &= 7 \cdot 5^{i+1} - 4 \cdot 3^i \\ &\leq 7 \cdot 5^{i+1} - 3 \cdot 3^i \\ &= 7 \cdot 5^{i+1} - 3^{i+1} \end{aligned}$$

ANSWER TO QUESTION 2. Let $c = 42$.

(a) Let $P(n)$ be the predicate "there are natural numbers a and b such that $n = 7 \cdot a + 8 \cdot b$" (in other words, $P(n)$ asserts that it is possible to form postage n using the available denominations of stamps: a represents the number of stamps of value 7, and b the number of stamps of value 8). We use (complete) induction to prove that $P(n)$ holds for every integer $n \geq 42$. (Note that we can also prove this using simple induction.)

Let i be an integer ≥ 42 such that for all j, $42 \leq j < i$, $P(j)$ holds. We wish to show that $P(i)$ holds.

CASE 1. $42 \leq i \leq 48$. (These are the base cases). We have:
$42 = 7 \cdot 6 + 8 \cdot 0$, $43 = 7 \cdot 5 + 8 \cdot 1$, $44 = 7 \cdot 4 + 8 \cdot 2$, $45 = 7 \cdot 3 + 8 \cdot 3$, $46 = 7 \cdot 2 + 8 \cdot 4$, $47 = 7 \cdot 1 + 8 \cdot 5$, $48 = 7 \cdot 0 + 8 \cdot 6$.

CASE 2. $i \geq 49$. (These are the induction cases.)
We have $42 \leq i-7 < i$. So by the induction hypothesis, we can choose a, b such that $i-7 = 7 \cdot a + 8 \cdot b$. So $i = 7 \cdot (a+1) + 8 \cdot b$, so $P(i)$ holds.

(b) We wish to show that it is not possible to form postage 41.
It is clear that we cannot use 6 or more stamps of value 8 since $8 \cdot 6 = 48 > 41$.
We also cannot use 5 or fewer stamps of value 8, since it is easy to check that for every integer b, $0 \leq b \leq 5$, it is the case that 7 does not divide $41 - 8 \cdot b$. So there is no way of forming postage 41.

ANSWER TO QUESTION 3. (We use $|S|$ to denote the number of elements in S.)

(a) Let $P(n)$ be:
"For every set S of n-bit strings such that no two strings in S differ in *exactly* one position, S contains no more than 2^{n-1} strings."
We use (simple) induction to prove that $P(n)$ holds for every integer $n \geq 1$. (Note that it is also possible to prove this without using induction.)

BASIS: We wish to show $P(i)$ holds for $i = 1$. So let S be a set of 1-bit strings, such that no two strings in S differ in exactly one position. Note that there are only two 1-bit strings, namely 0 and 1. Since these strings differ in exactly one position, S cannot contain both of them. So $|S| \leq 1 = 2^{i-1}$.

INDUCTION STEP: Let $i \geq 1$ be an integer such that $P(i)$ holds. We wish to show that $P(i+1)$ holds.

So let S be a set of $(i+1)$-bit strings, such that no two strings in S differ in exactly one position. We wish to show that $|S| \leq 2^i$. Let S_0 be the set of strings in S that begin with 0, and let S_1 be the set of strings in S that begin with 1. Since every string from S is in one of S_0 or S_1 but not both, we have $|S| = |S_0| + |S_1|$.

We now claim that $|S_0| \leq 2^{i-1}$. To see this, define the set of i-bit strings $S_0' = \{\, x \mid 0x \text{ is in } S_0\}$; that is, S_0' is the set of strings obtained by chopping off the first 0 from all strings in S_0. No two strings in S_0' differ in exactly one position (since if $x, y \in S_0'$ differed in exactly one position, then $0x$ and $0y$ would be two strings in S that differ in exactly one position). So by the induction hypothesis $P(i)$, $|S_0'| \leq 2^{i-1}$. Since $|S_0| = |S_0'|$, we have $|S_0| \leq 2^{i-1}$.

One can prove in a similar manner that $|S_1| \leq 2^{i-1}$. So $|S| = |S_0| + |S_1| \leq 2^{i-1} + 2^{i-1} = 2^i$.

(b) – Extra Credit: Let n be a positive integer. Let S be the set of n-bit strings x such that the bit 0 occurs in x an *even* number of times.

Two see that no two strings in S differ in exactly one position, consider two strings $x, y \in S$. Say that x and y differ in exactly one position. Then x and y are exactly the same, except that in one position x has a 0 and y has a 1. So 0 occurs in x exactly one more time than in y, contradicting the fact that both x and y are in S.

We also claim that $|S| = 2^{n-1}$. Although one can use induction to prove this, here is another way. Let T be the set of n-bit strings y such that the bit 0 occurs in y an *odd* number of times. Define the function f on S by $f(x) =$ the string obtained by "flipping" the first bit of x. One can show that f maps S one-one, onto T, and so $|S| = |T|$. Since each of the 2^n n-bit strings belongs to exactly one of S or T, we have $2^n = |S| + |T| = 2|S|$, so $|S| = 2^{n-1}$.

ANSWER TO QUESTION 4. Let $P(n)$ be the predicate: $a_n < 2^n$.
We use (complete) induction to prove that $P(n)$ holds for every integer $n \geq 2$.

Let i be an integer ≥ 2 such that for all j, $2 \leq j < i$, $P(j)$ holds. We wish to show that $P(i)$ holds.

Case 1. $2 \leq i \leq 4$. (These are the base cases). We have:
$a_2 = 2 < 2^2$, $a_3 = 2 + 2 + 2 = 6 < 2^3$, $a_4 = 6 + 2 + 2 = 10 < 2^4$.

Case 2. $i \geq 5$. (These are the induction cases.)
Each of the numbers $i - 1$, $i - 2$ and $i - 3$ are $< i$, and since $i \geq 5$, each of them are also ≥ 2. So by applying the induction hypothesis to each of them, we have:
$a_{i-1} < 2^{i-1}$, $a_{i-2} < 2^{i-2}$ and $a_{i-3} < 2^{i-3}$.
Since $i \geq 3$, we have $a_i = a_{i-1} + a_{i-2} + a_{i-3}$.
So $a_i < 2^{i-1} + 2^{i-2} + 2^{i-3} = (4 + 2 + 1) \cdot 2^{i-3} < 8 \cdot 2^{i-3} = 2^i$.

ANSWER TO QUESTION 5. Let $x \in \mathbb{N}$. In order to prove termination of the program, we first prove the following loop invariant lemma.

Lemma: After every iteration of the loop (including the "zero-th"), $y = x^2$ and $x \geq 0$.

Before proving the Lemma, we show how to use it to prove termination of the program. Clearly x is an integer quantity that decreases each time through the loop, and the Lemma tells us that x is bounded below by 0. By the Termination Principle, the program halts.

We now prove the Lemma. For every $n \in \mathbb{N}$ such that the loop is executed at least n times, let x_n and y_n be the values of x and y after n iterations.

Let $P(n)$ be the predicate: "If the loop is executed at least n times, $y_n = x_n^2$ and $x_n \geq 0$". We will prove that $P(n)$ holds for all $n \in \mathbb{N}$, by (simple) induction.

Basis: Clearly (by the program) $y_0 = x_0^2$, and we have already assumed that $x_0 \geq 0$.

Induction Step: Let i be an arbitrary natural number and suppose that $P(i)$ holds. We'll show that $P(i + 1)$ holds as well.

Assume that the loop is executed at least $i + 1$ times. By the induction hypothesis, we have $y_i = x_i^2$ and $x_i \geq 0$. By the program, $x_{i+1} = x_i - 1$ and $y_{i+1} = y_i - 2x_{i+1} - 1$. So
$y_{i+1} = y_i - 2x_{i+1} - 1 = x_i^2 - 2(x_i - 1) - 1 = x_i^2 - 2x_i + 1 = (x_i - 1)^2 = x_{i+1}^2$.

Also, since the loop is executed at least $i + 1$ times, we must have $y_i \neq 0$, and so $x_i^2 \neq 0$, and so $x_i \neq 0$. Since $x_i \geq 0$, we have $x_i > 0$. So $x_{i+1} = x_i - 1 \geq 0$.

QUESTION: (left as an exercise)
Since all we needed to prove termination was the loop invariant "$x \geq 0$", why did we bother to prove as well the invariant $y = x^2$?
