# CSC165 Tutorial #6

Sample Solutions

Winter 2015

Work on these exercises *before* the tutorial. You don't have to come up with complete solutions before the tutorial, but you should be prepared to discuss them with your TA.

IMPORTANT: You **must** use the proof structures and format of this course.

Prove or disprove each of the following bounds.
In all questions, assume that $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$.

1. Let $f(n) = \frac{1}{5}n^2 - 30n - 5$, and $g(n) = n^2$. Then $f \in \Omega(g)$.

   Since this question gave many of you grief, let's walk through one of the ways in which you can tackle it.

   **Rough notes:**

   (a) Since the highest degree of either $f$ or $g$ is 2, you should intuitively understand that $f \in \Omega(g)$. Therefore, you need to find a way to prove that $f \geq cg(n)$, or $\frac{1}{5}n^2 - 30n - 5 \geq cn^2$.

   (b) Try to find a way to manipulate $f(n)$, i.e. to find a method for $\frac{1}{5}n^2 - 30n - 5 \geq ... \geq .... \geq cn^2$. Note that you shouldn't feel pressured to choose your $c'$ and $B'$ at this stage; you should be able to see what $c'$ should be after you have a rough idea of how you'll arrive from $f$ to $g$, and you should choose $B'$ when you figure out that your proof won't work for certain values of $n$.

   (c) You may get stuck while doing the above; we certainly did get stuck in tutorial.

   (d) Let's back away from trying to algebraically solve this. You probably noticed by now that $-30n-5$ is quite problematic. Remember than $f$ should somehow be greater than $c \cdot g$, but that $c$ must be positive; however, $f$ is negative for quite some time. In fact, use the quadratic formula to figure out when it even becomes positive for $n \in \mathbb{N}$.

   $$\begin{aligned} n &= \frac{30 \pm \sqrt{(-30)^2 - 4(\frac{1}{5})(-5)}}{2(\frac{1}{5})} \\ &= \frac{150 \pm 5\sqrt{904}}{2} \\ &\doteq 75 \pm 75 \end{aligned}$$

   Note that the precise value isn't important. All you need to know is that before around $n = 150$, $f(n)$ can't be positive (since $n$ is a natural number). This is important for two reasons:

   i. Recall that the question assumes that $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, not just $\mathbb{R}$

   ii. Recall that $c$ must be positive, and if $f$ is negative, since $g$ is by definition positive, $f$ must have a value of at least zero or greater.

   This tells you that your $B'$ must be at least 151. However, at $n = 151$, $f(n)$ is just about 25, whereas $g(n)$ is over 22,000. Does it ever become bigger?

(e) The above can be easily resolved if we just pick $c'$ small enough, say, 0.00000000000001. But, you can't just pick a specific value, since I can always use a much bigger value for $n$, such that $c \cdot g(n)$ will again be larger than $f(n)$.

(f) Your goal now should be to find a way to pick a small enough $c'$ so you can finally get over this proof and move on with your life. If you're stuck, sit back and think about all the things you've learnt in this course. By the way, this strategy works for almost everything in school.

(g) Hint: our antecedent in complexity proofs ($n \geq B', n \in \mathbb{N}$) is for any generic value of $n$. However, even though it's for "any" $n$, we can still restrict it by making it equal to or greater than a $B'$ of our choosing. Now, we can't tie our value of $c'$ to $n^*$, but we can tie it to another value that changes based on the circumstances. So, what if we use a similar concept for choosing our $c$, but backwards—have a value for $c'$ that depends on a changing variable?

Try it yourself. There's another hint on the next page, and the solution on the page after.

*Why not? Think about the order in which you introduce variables. If you say $c' = n$, what is $n$? If you assume $n \geq B', n \in \mathbb{N}$ before you choose a $c'$, what happens? Think about the order of quantifiers and how the meaning of the statement changes as you switch their order.

(h) Hint: Q4 of this tutorial exercise uses the concept of limits to prove the statement. Can we use it here? Try it yourself before reading further.

(i) Calculate the limit of ${}^{f(n)}/_{g(n)}$ as $n \to \infty$.

$$\lim_{n\to\infty} \frac{\frac{1}{5}n^2 - 30n - 5}{n^2} = \frac{1}{5}$$

Since the limit is a value other than 0 or $\infty$, we can use the following definition:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L \iff \forall \epsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \implies L - \epsilon < \frac{f(n)}{g(n)} < L + \epsilon$$

In our case, since $\lim_{n\to\infty} \frac{\frac{1}{5}n^2-30n-5}{n^2} = \frac{1}{5}$, we have the following as fact:

$$\forall \epsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \implies \boxed{\frac{1}{5} - \epsilon < \frac{f(n)}{g(n)}} < \frac{1}{5} + \epsilon$$

Looks like we found a way to make $c'$ very small! Just like in a regular proof involving limits, we just need to make sure that we can claim $n \geq n_0$ as above, in good faith.

The full proof is below.

**Solution:**

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{5}$, we know that $\forall \epsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \implies \frac{1}{5} - \epsilon < \frac{f(n)}{g(n)} < \frac{1}{5} + \epsilon$

Let $c' = \frac{1}{5} - \epsilon$, such that the definition above is true, and such that $c' \in \mathbb{R}^+$

// Note that there's no need to clarify the domain for $\epsilon$, since $\epsilon, c \in \mathbb{R}^+$ already restricts it

Let $n_0$ be such that $\forall \epsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \implies \frac{1}{5} - \epsilon < \frac{f(n)}{g(n)} < \frac{1}{5} + \epsilon$

  Let $B' = \max(151, n_0)$; then $B' \in \mathbb{N}$   # so that $f(n) > 0$

  // We established that $n$ must be at least 151 for $f(n)$ to be positive;

  // it can be larger (if $n_0 > 151$), but it can't be smaller

    Assume $n \geq B', n \in \mathbb{N}$   # antecedent

      Then $n \geq n_0$   # by definition of max

      Then $\frac{1}{5} - \epsilon < \frac{f(n)}{g(n)} < \frac{1}{5} + \epsilon$   # from the limit definition

      Then $\frac{1}{5} - \epsilon < \frac{f(n)}{g(n)}$

      Then $(\frac{1}{5} - \epsilon) \cdot g(n) < f(n)$

      Then $f(n) \geq (\frac{1}{5} - \epsilon)g(n) = c'g(n)$   # $> \implies \geq$ (but not vice versa!)

    Then $\forall n \in \mathbb{N}, n \geq B' \implies f(n) \geq c'g(n)$   # complete the assumption

  Then $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies f(n) \geq c'g(n)$   # complete the assumption

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies f(n) \geq cg(n)$   # complete the assumption

So, $f \in \Omega(g)$.

2. Let $f(n) = \sqrt{n}(40n^3 + 6)$, and $g(n) = n^{7/2}$. Then $f \in \mathcal{O}(g)$.

**Solution:**

Let $c' = 46$; then $c' \in \mathbb{R}^+$

  Let $B' = 0$; then $B' \in \mathbb{N}$

    Assume $n \geq B', n \in \mathbb{N}$   # antecedent

      Then $f(n) = 40n^{7/2} + 6n^{1/2} \leq 40n^{7/2} + 6n^{7/2}$   # $n \in \mathbb{N}$

      $\leq 46n^{7/2} = c'g(n)$   # you should choose $c'$ and $B'$ around here

    Then $\forall n \in \mathbb{N}, n \geq B' \implies f(n) \leq c'g(n)$   # complete the assumption

  Then $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies f(n) \leq c'g(n)$   # complete the assumption

Then $\exists c \in \mathcal{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies f(n) \leq cg(n)$   # complete the assumption

So, $f \in \mathcal{O}(g)$.

3. Let $f(n) = \max(n^2, 100)(3n+1) - 5$, and $g(n) = n^3$. Then $f \in \Theta(g)$.

**Solution:**

Let $c_1 = 1$ and $c_2 = 395$; then $c_1, c_2 \in \mathbb{R}^+$
  Let $B' = 1$; then $B' \in \mathbb{N}$
    Assume $n \geq B', n \in \mathbb{N}$
      Case 1: $\max(n^2, 100) = 100$; then $n \leq 10$
        // Proving $f \in \Omega(g)$
        Then $f(n) = 300n + 100 - 5 = 300n + 95$

$$\begin{aligned} &\geq 300n \quad \# \ 95 \in \mathbb{N} \\ &\geq (n^2)n \quad \# \ n^2 \leq 100 < 300, n \leq 10 \\ &= 1 \cdot n^3 = c_1 g(n) \end{aligned}$$

        Then $f \in \Omega(g)$
        // Proving $f \in \mathcal{O}(g)$
        Then $f(n) = 300n + 100 - 5 = 300n + 95$

$$\begin{aligned} &\leq 300n + 95n \quad \# \ n \in \mathbb{N}, n \geq 1 \\ &\leq 395n^3 \quad \# \ n \in \mathbb{N}, n^3 > n \\ &= 395 \cdot n^3 = c_2 g(n) \end{aligned}$$

        Then $f \in \mathcal{O}(g)$
        Since $f \in \Omega(g)$ and $f \in \mathcal{O}(g)$, $f \in \Theta(g)$
      Case 2: $\max(n^2, 100) = n^2$; then $n > 10$
        // Proving $f \in \Omega(g)$
        Then $f(n) = 3n^3 + n^2 - 5$

$$\begin{aligned} &\geq 3n^3 - 5 \quad \# \ n^2 \in \mathbb{N} \\ &= n^3 + 2n^3 - 5 \\ &\geq 1 \cdot n^3 = c_1 g(n) \quad \# \text{ for } n > 10, 2n^3 - 5 > 0 \end{aligned}$$

        Then $f \in \Omega(g)$
        // Proving $f \in \mathcal{O}(g)$
        Then $f(n) = 3n^3 + n^2 - 5$

$$\begin{aligned} &\leq 3n^3 + n^2 \quad \# \ 5 \in \mathbb{N} \\ &\leq 3n^3 + n^3 = 4n^3 \quad \# \ n \in \mathbb{N} \\ &\leq 395 \cdot n^3 = c_1 g(n) \end{aligned}$$

        Then $f \in \mathcal{O}(g)$
        Since $f \in \Omega(g)$ and $f \in \mathcal{O}(g)$, $f \in \Theta(g)$
      Then $f \in \Theta(g)$
    Then $\forall n \in \mathbb{N}, n \geq B' \implies c_1 g(n) \leq f(n) \leq c_2 g(n)$
  Then $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies c_1 g(n) \leq f(n) \leq c_2 g(n)$
Then $\exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies c_1 g(n) \leq f(n) \leq c_2 g(n)$

Yes, in the real world we wouldn't worry about values of $n \leq 10$. However, this proof is exhaustive, and therefore you must cover the possibility, since for such low values you have an entirely different function for the runtime of this algorithm.

4. Let $f(n) = \mid n^2 - n^5 - 2n + 6 \mid$, and $g(n) = n^2$. Then $f \in \mathcal{O}(g)$.

**Solution:**

You should intuitively know that this is false, because of the highest order of $f$ vs. $g$. Therefore, you will prove the negation: $f \notin \mathcal{O}(g) \iff \forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$. We will use the limit definition for this proof.

$$lim_{n\to\infty} \frac{\mid n^2 - n^5 - 2n + 6 \mid}{n^2} = \infty$$

Therefore, we have a fact: $\forall \epsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \implies \frac{f(n)}{g(n)} > \epsilon$.

Assume $c \in \mathbb{R}^+, B \in \mathbb{N}$
We know that $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies \frac{f(n)}{g(n)} > c$ # from definition, with $\epsilon = c$
// Prove the first part of your statement, *i.e.* $n \geq B$
Let $n_1$ be such that $\forall n \in \mathbb{N}, n \geq n_1 \implies \frac{|n^2-n^5-2n+6|}{n^2} > c$
Let $n_0 = \max(B, n_1)$; then $n_0 \in \mathbb{N}$
// Since you need to show both that $n \geq B$ (for your proof statement) and that $n \geq n_1$ (for
// your limit consequent), you need to ensure that $n$ can satisfy both; so, you don't need to
// pick any specific value, just the largest of the two
Then, $n_0 \geq B$ # definition of max
// Now, prove that $f(n) > cg(n)$
Since $n_0 \geq n_1$, $\frac{|n_0^2-n_0^5-2n_0+6|}{n_0^2} > c$ # follows from limit definition
Then $\mid n_0^2 - n_0^5 - 2n_0 + 6 \mid > cn_0^2$, and so $f(n_0) > cg(n_0)$
Then, $n_0 \geq B \wedge f(n_0) > cg(n_0)$ # simple conjunction
Then $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$ # note that the $n$ is $n_0$ in the proof
So, $f \notin \mathcal{O}(g)$