CSC165 Tutorial #5

Exercises

Winter 2015

Here are three general steps that might help you start and complete a proof:

1. Start by translating the claim into the logical notation. This will help you to see the logical structure of the claim (e.g. whether it's universally or existentially quantified, if it includes an implication, etc.) and identify appropriate proof structures.

2. Write the outline of the proof based on the logical structure of the claim. The outline must be in the proof format of this course and include all assumptions and conclusions that can be derived solely based on the claim.

   For example if the translation of the claim looks like: $\forall x \in D, P(x) \Rightarrow (Q(x) \wedge R(x))$, the outline of the proof should look like the following:

   Assume $x \in D$. # x is a typical element of D
       Assume $P(x)$. # antecedent
           $\vdots$
           Then $Q(x)$. # ...
           $\vdots$
           Then $R(x)$. # ...
           Then $Q(x) \wedge R(x)$. # introduce $\wedge$
       Then $P(x) \Rightarrow (Q(x) \wedge R(x))$. # introduce $\Rightarrow$
   Then $\forall x \in D, P(x) \Rightarrow (Q(x) \wedge R(x))$. # introduce $\forall$

3. Try to fill in the outline (i.e. the "...") by appropriate derivations/arguments: this step is usually the hardest part of a proof. It requires mathematical creativity and might take some time. Think of it as solving a puzzle! But instead of using common-sense knowledge, you are only allowed to use mathematical facts.

   Here are some tips that might be helpful:

   - Start by doing some **scratch work** (outside the outline) to figure out how the proof can be done. The problem-solving techniques (presented in the first lecture) might be helpful here. For example, check the claim for *some specific examples* to convince yourself why it's true.
   - You might also want to prove the claim only for 1-2 *specific elements of the domain* first to develop some **intuition** about the direction of the proof, and then use the intuition to *generalize* the proof for all elements of the domain.
   - If the question provides some definitions/assumptions, translate them into symbolic notation. Then try to manipulate the definitions/assumptions until you derived the claim.
   - In the manipulation process, you should use the mathematical properties of the domains, functions and predicates in the claim. So, it might be helpful if you **list** (in symbolic form) all the (relevant) facts about the domains, functions and predicates, and then see how they can be used in the proof.

For all questions, try to **complete** Steps (1) and (2) (described in the previous page) before the tutorial session, and have some **scratch work** for Step (3).

IMPORTANT: You **must** use the proof structures and format of this course.

1. Use **proof by contradiction** to show that there is no integer that is both even and odd.

2. Prove or disprove the following statement:

   $\mathbf{S_1}$: If product of two positive real numbers is greater than 50, then at least one of the numbers is greater than 7.

3. Prove the following statement:

   $\mathbf{S_2}$: If product of two positive real numbers is greater than $z \in \mathbb{R}^+$, then at least one of the numbers is greater than $\sqrt{z}$.

   Hint: Note that $\mathbf{S_1}$ is a special case of $\mathbf{S_2}$.

4. Recall that for integers x, y, z, the notation $x \equiv y \bmod z$ means "$x - y$ is a multiple of z."

   Use this definition to prove the following statement.

   $$\forall n \in \mathbb{N}, (n^3 - n) \equiv 0 \bmod 6.$$

   Hint 1: Recall that $n \in \mathbb{N}$ is a multiple of 6 if and only if n is multiple of both 2 and 3.

   Hint 2: Recall that $(n^2 - 1) = (n - 1)(n + 1)$.