CSC165 Tutorial #5

Sample Solutions

Winter 2015

1. Use **proof by contradiction** to show that there is no integer that is both even and odd.

Solution:

(1) Translating the claim: Here's the translation

$$\neg(\exists n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k) \wedge (\exists k \in \mathbb{Z}, n = 2k + 1)).$$

(2) Deriving the outline of the proof: Since we are asked to prove the claim by contradiction, we must assume the negation of the claim, which is the following statement, and try to derive a contradiction:

$$\exists n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k) \wedge (\exists k \in \mathbb{Z}, n = 2k + 1).$$

The proof outline should look like the following:

Assume $\exists n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k) \wedge (\exists k \in \mathbb{Z}, n = 2k + 1)$. # to derive contradiction
  Let $k_0, k_1, n_0 \in \mathbb{Z}$ such that $n_0 = 2k_0$ and $n_0 = 2k_1 + 1$. # instantiate $\exists$
    ⋮
  Contradiction! # ...
Then $\neg(\exists n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k) \wedge (\exists k \in \mathbb{Z}, n = 2k + 1))$. # assuming the negation leads to a contradiction

(3) Scratch Work: Based on the outline, there should be three integers k_0, k_1, n_0 such that $n_0 = 2k_0$ and $n_0 = 2k_1 + 1$.

What do these two expressions tell you?

The first thing that comes to mind is that $2k_0 = 2k_1 + 1$. Try to see if it gives you a contradiction!

From $2k_0 = 2k_1 + 1$ we can conclude that $2k_0 - 2k_1 = 1$, and so $k_0 - k_1 = 1/2$. Do you see a contradiction?

(4) Putting everything together: Once you have a sketch of the proof, you can put it in the proof structure

Assume $\exists n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k) \wedge (\exists k \in \mathbb{Z}, n = 2k + 1)$. # to derive contradiction
  Let $k_0, k_1, n_0 \in \mathbb{Z}$ such that $n_0 = 2k_0$ and $n_0 = 2k_1 + 1$. # instantiate $\exists$
  Then $n_0 - n_0 = 0 = 2(k_0 - k_1) - 1$. # algebra
  Then $k_0 - k_1 = 1/2$. # algebra
  Contradiction! # $k_0, k_1 \in \mathbb{Z}$ and their deference must be in $\mathbb{Z}$
Then $\neg(\exists n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k) \wedge (\exists k \in \mathbb{Z}, n = 2k + 1))$. # assuming the negation leads to a contradiction

2. Prove or disprove the following statement:

$\mathbf{S_1}$: If product of two positive real numbers is greater than 50, then at least one of the numbers is greater than 7.

Solution:

(1) Translating the claim: Here's the translation

$$\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, ((x.y) > 50) \Rightarrow (x > 7) \vee (y > 7).$$

(2) Deriving the outline of the proof: First, you need to decide if the claim is true or false. The contrapositive of the implication seems rather obvious, but if you don't see it immediately you can verify the claim for some specific examples.

From the translation we can see that the claim is a universally quantified implication. Since proving the contrapositive seems to be easier, we will use the indirect proof structure. So the proof outline should look like the following:

Assume $x, y \in \mathbb{R}^+$. # x, y are typical positive real numbers
  Assume $(x \leq 7) \wedge (y \leq 7)$. # antecedent of the contrapositive
    ⋮
    Then $(x.y) \leq 50$. # ...
  Then $(x \leq 7) \wedge (y \leq 7) \Rightarrow (x.y) \leq 50$. # introduce $\Rightarrow$
  Then $((x.y) > 50) \Rightarrow (x > 7) \vee (y > 7)$. # contrapos. is equivalent to the implication
Then $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, ((x.y) > 50) \Rightarrow (x > 7) \vee (y > 7)$. # introduce $\forall$

(3) Scratch Work: Now you only need to fill in the "..." in the proof outline.

Take a look at the assumptions that you have: $0 < x \leq 7$ and $0 < y \leq 7$.

It's not hard to see that $(x.y) \leq 49$, and therefore $(x.y) \leq 50$.

(4) Putting everything together:

Assume $x, y \in \mathbb{R}^+$. # x, y are typical positive real numbers
  Assume $(x \leq 7) \wedge (y \leq 7)$. # antecedent of the contrapositive
    Then $(x.y) \leq 49 \leq 50$. # since $0 < x \leq 7$ and $0 < y \leq 7$
  Then $(x \leq 7) \wedge (y \leq 7) \Rightarrow (x.y) \leq 50$. # introduce $\Rightarrow$
  Then $((x.y) > 50) \Rightarrow (x > 7) \vee (y > 7)$. # contrapos. is equivalent to the implication
Then $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, ((x.y) > 50) \Rightarrow (x > 7) \vee (y > 7)$. # introduce $\forall$

Alternative Solution: It's also possible to prove the claim by contradiction (recall that indirect proof is a special case of proof by contradiction).

The structure of the proof will be different, but the arguments are similar.

Assume $\exists x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, ((x.y) > 50) \wedge (x \leq 7) \wedge (y \leq 7)$. # to derive contradiction
  Let $x_0, y_0 \in \mathbb{R}^+$ such that $x_0 \leq 7$ and $y_0 \leq 7$ and $(x_0.y_0) > 50$. # instantiate $\exists$
  Then $(x_0.y_0) \leq 49$. # since $0 < x_0 \leq 7$ and $0 < y_0 \leq 7$
  Contradiction! # by assumption $(x_0.y_0) > 50$
Then $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, ((x.y) > 50) \Rightarrow (x > 7) \vee (y > 7)$. # assuming the negation leads to a contradiction

3. Prove or disprove the following statement:

   $\mathbf{S_2}$: If product of two positive real numbers is greater than $z \in \mathbb{R}^+$, then at least one of the numbers is greater than $\sqrt{z}$.

   Hint: Note that $\mathbf{S_1}$ is a special case of $\mathbf{S_2}$.

   Solution:

   (1) Translating the claim: Here's the translation

$$\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, \forall z \in \mathbb{R}^+, ((x.y) > z) \Rightarrow (x > \sqrt{z}) \vee (y > \sqrt{z}).$$

   (2) Deriving the outline of the proof: First, you need to decide if the claim is true or false.

   You can see that the claim generalizes $\mathbf{S_1}$. Again, it's easy to see that the contrapositive of the implication is true.

   We will also have a similar outline:

   Assume $x, y, z \in \mathbb{R}^+$. # x, y, z are typical positive real numbers
   Assume $(x \leq \sqrt{z}) \wedge (y \leq \sqrt{z})$. # antecedent of the contrapositive
   ⋮
   Then $(x.y) \leq z$. # ...
   Then $(x \leq \sqrt{z}) \wedge (y \leq \sqrt{z}) \Rightarrow (x.y) \leq z$. # introduce $\Rightarrow$
   Then $((x.y) > z) \Rightarrow (x > \sqrt{z}) \vee (y > \sqrt{z})$. # contrapos. is equivalent to the implication
   Then $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, \forall z \in \mathbb{R}^+, ((x.y) > z) \Rightarrow (x > \sqrt{z}) \vee (y > \sqrt{z})$. # introduce $\forall$

   (3) Scratch Work: The proof sketch is also very similar.

   By the assumptions we have: $0 < x \leq \sqrt{z}$ and $0 < y \leq \sqrt{z}$.

   Thus, $(x.y) \leq (\sqrt{z}.\sqrt{z}) = z$.

   (4) Putting everything together:

   Assume $x, y, z \in \mathbb{R}^+$. # x, y, z are typical positive real numbers
   Assume $(x \leq \sqrt{z}) \wedge (y \leq \sqrt{z})$. # antecedent of the contrapositive
   Then $(x.y) \leq (\sqrt{z}.\sqrt{z}) = z$. # since $0 < x \leq \sqrt{z}$ and $0 < y \leq \sqrt{z}$
   Then $(x \leq \sqrt{z}) \wedge (y \leq \sqrt{z}) \Rightarrow (x.y) \leq z$. # introduce $\Rightarrow$
   Then $((x.y) > z) \Rightarrow (x > \sqrt{z}) \vee (y > \sqrt{z})$. # contrapos. is equivalent to the implication
   Then $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, \forall z \in \mathbb{R}^+, ((x.y) > z) \Rightarrow (x > \sqrt{z}) \vee (y > \sqrt{z})$. # introduce $\forall$

   Alternative Solution: We can also prove the claim by contradiction

   Assume $\exists x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, \exists z \in \mathbb{R}^+, ((x.y) > z) \wedge (x \leq \sqrt{z}) \wedge (y \leq \sqrt{z})$. # to derive contradiction
   Let $x_0, y_0, z_0 \in \mathbb{R}^+$ such that $x_0 \leq \sqrt{z_0}$ and $y_0 \leq \sqrt{z_0}$ and $(x_0.y_0) > z_0$. # instantiate $\exists$
   Then $(x_0.y_0) \leq z_0$. # since $0 < x_0 \leq \sqrt{z_0}$ and $0 < y_0 \leq \sqrt{z_0}$
   Contradiction! # by assumption $(x_0.y_0) > z_0$
   Then $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+, \forall z \in \mathbb{R}^+, ((x.y) > z) \Rightarrow (x > \sqrt{z}) \vee (y > \sqrt{z})$. # assuming the negation leads to a contradiction

4. Recall that for integers x, y, z, the notation $x \equiv y \bmod z$ means "$x - y$ is a multiple of z."
   Use this definition to prove the following statement.

$$\forall n \in \mathbb{N}, (n^3 - n) \equiv 0 \bmod 6.$$

   Hint 1: Recall that $n \in \mathbb{N}$ is a multiple of 6 if and only if n is multiple of both 2 and 3.

   Hint 2: Recall that $(n^2 - 1) = (n - 1)(n + 1)$.

Solution:

(1) Translating the claim: The claim is already in logical notation. But it might be helpful if we re-state it into a simpler statement using the definition of *mod*.

The statement says that $(n^3 - n) - 0 = (n^3 - n)$ is a multiple of 6. So we can re-state it as the following

$$\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, (n^3 - n) = 6k.$$

(2) Deriving the outline of the proof: We have a universally quantified statement, but the given assumption (i.e. $n \in \mathbb{N}$) alone does not seem to be enough for proving that $(n^3 - n)$ is a multiple of 6; we should consider other properties of natural numbers!

Since the claim is about multiples of 6, it seems relevant that we consider prime factors of 6: we can show that $(n^3 - n)$ is a multiple both 2 and 3, and then conclude that that it's a multiple of 6. (Note that by doing so we are breaking the claim into the statements that are simpler to prove)

To prove that $(n^3 - n)$ is a multiple of 2, we consider two cases: (1) n is odd (2) n is even.

To prove that $(n^3 - n)$ is a multiple of 3, we consider three cases: (1) n is a multiple of 3 (2) the reminder of n divided by 3 is 1 (3) the reminder of n divided by 3 is 2.

So the outline should look like the following:

Assume $n \in \mathbb{N}$. # n is a typical natural number
  Case 1: Assume exists $k_0 \in \mathbb{N}$ such that $n = 2k_0$
    ⋮
    Then exists $k_1 \in \mathbb{N}$ such that $n^3 - n = 2k_1$. # ...
  Case 2: Assume exists $k_0 \in \mathbb{N}$ such that $n = 2k_0 + 1$
    ⋮
    Then exists $k_1 \in \mathbb{N}$ such that $n^3 - n = 2k_1$. # ...
  Then exists $k_1 \in \mathbb{N}$ such that $n^3 - n = 2k_1$. # true for both cases
  Case 3: Assume exists $k_0 \in \mathbb{N}$ such that $n = 3k_0$
    ⋮
    Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # ...
  Case 4: Assume exists $k_0 \in \mathbb{N}$ such that $n = 3k_0 + 1$
    ⋮
    Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # ...
  Case 5: Assume exists $k_0 \in \mathbb{N}$ such that $n = 3k_0 + 2$
    ⋮
    Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # ...
  Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # true for all three cases
  Then exists $k_1 \in \mathbb{N}$ s.t. $n^3 - n = 2k_1$ and $k_2 \in \mathbb{N}$ s.t. $n^3 - n = 3k_2$.# proved that both are true
  Then exists $k \in \mathbb{N}$ such that $n^3 - n = 6k$. # both 2 and 3 are factors of $n^3 - n$, so $2 * 3 = 6$ is a factor of $n^3 - n$
Then $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n^3 - n = 6k$. # introduce $\forall$ and $\exists$
Then $\forall n \in \mathbb{N}, (n^3 - n) \equiv 0 \bmod 6$. # by definition of $\equiv$ mod

(3) Scratch Work: We want to show that $n^3 - n$ is a multiple of both 2 and 3.

By factorizing $n^3 - n$ we will get: $n^3 - n = n(n-1)(n+1)$.

It's not hard to see that $n^3 - n$ is always even (either n is even or $n+1$ is even).

Using the definition of multiples of 3, and by doing some algebra, you can also show that either $n-1$ is a multiple of 3, or n is a multiple of 3, or $n+1$ is a multiple of 3.

(4) Putting everything together:

Assume $n \in \mathbb{N}$. # n is a typical natural number
  Then $n^3 - n = n(n^2-1) = n(n-1)(n+1)$. # algebra
  Case 1: Assume exists $k_0 \in \mathbb{N}$ such that $n = 2k_0$
    Then $n^3 - n = 2k_0(2k_0-1)(2k_0+1)$. # substitute n by $2k_0$
    Then exists $k_1 \in \mathbb{N}$ such that $n^3 - n = 2k_1$. # $k_1 = k_0(2k_0-1)(2k_0+1)$ and $k_1 \in \mathbb{N}$
  Case 2: Assume exists $k_0 \in \mathbb{N}$ such that $n = 2k_0 + 1$
    Then $n^3 - n = (2k_0+1)2k_0(2k_0+2)$. # substitute n by $2k_0 + 1$
    Then exists $k_1 \in \mathbb{N}$ such that $n^3 - n = 2k_1$. # $k_1 = (2k_0+1)k_0(2k_0+2)$ and $k_1 \in \mathbb{N}$
  Then exists $k_1 \in \mathbb{N}$ such that $n^3 - n = 2k_1$. # true for both cases
  Case 3: Assume exists $k_0 \in \mathbb{N}$ such that $n = 3k_0$
    Then $n^3 - n = 3k_0(3k_0-1)(3k_0+1)$. # substitute n by $3k_0$
    Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # $k_2 = k_0(3k_0-1)(3k_0+1)$ and $k_2 \in \mathbb{N}$
  Case 4: Assume exists $k_0 \in \mathbb{N}$ such that $n = 3k_0 + 1$
    Then $n^3 - n = (3k_0+1)3k_0(3k_0+2)$. # substitute n by $3k_0 + 1$
    Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # $k_2 = (3k_0+1)k_0(3k_0+2)$ and $k_2 \in \mathbb{N}$
  Case 5: Assume exists $k_0 \in \mathbb{N}$ such that $n = 3k_0 + 2$
    Then $n^3 - n = (3k_0+2)(3k_0+1)(3k_0+3)$. # substitute n by $3k_0 + 2$
    Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # $k_2 = (3k_0+2)(3k_0+1)(k_0+1)$ and $k_2 \in \mathbb{N}$
  Then exists $k_2 \in \mathbb{N}$ such that $n^3 - n = 3k_2$. # true for all three cases
  Then exists $k_1 \in \mathbb{N}$ s.t. $n^3 - n = 2k_1$ and $k_2 \in \mathbb{N}$ s.t. $n^3 - n = 3k_2$.# proved that both are true
  Then exists $k \in \mathbb{N}$ such that $n^3 - n = 6k$. # both 2 and 3 are factors of $n^3 - n$, so $2 * 3 = 6$ is a factor of $n^3 - n$
Then $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, n^3 - n = 6k$. # introduce $\forall$ and $\exists$
Then $\forall n \in \mathbb{N}, (n^3 - n) \equiv 0 \bmod 6$. # by definition of $\equiv$ mod