CSC165 Tutorial #4

Sample Solutions

Winter 2015

Work on these exercises *before* the tutorial. You don't have to come up with complete solutions before the tutorial, but you should be prepared to discuss them with your TA.

1. Use the proof structure of this course to **disprove** claim S_1:

$$S1: \qquad \forall x \in \mathbb{R}, 2\lfloor x \rfloor > 2x - 1$$

Solution: We prove the negation of S_1:

$$\neg S_1: \qquad \exists x \in \mathbb{R}, 2\lfloor x \rfloor \leq 2x - 1$$

Let $x = 0.6$. Then $x \in \mathbb{R}$. # $0.6 \in \mathbb{R}$
Then $2\lfloor x \rfloor = 2\lfloor 0.6 \rfloor = 2 \cdot 0 = 0$ # 0 is greatest integer ≤ 0.6
Then $2x - 1 = 2(0.6) - 1 = 0.2 \geq 0 = 2\lfloor x \rfloor$ # by choice of x
Then $\exists x \in \mathbb{R}, 2\lfloor x \rfloor \leq 2x - 1$ # introduced $\exists$

2. Consider the definitions:

$$\forall n \in \mathbb{N}, U(n) \Leftrightarrow \exists q \in \mathbb{N}, n = 5q + 3$$
$$\forall m \in \mathbb{N}, V(m) \Leftrightarrow \exists q' \in \mathbb{N}, m = 5q' + 4$$
$$\forall p \in \mathbb{N}, W(p) \Leftrightarrow \exists q'' \in \mathbb{N}, p = 5q'' + 2$$

Use the definitions and the proof structure of this course to prove statement S_2:

$$S_2 : \qquad \forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (V(m) \wedge U(n)) \Rightarrow W(m \times n)$$

Solution:

Assume $m \in \mathbb{N}$, assume $n \in \mathbb{N}$. # m, n are generic natural numbers
  Assume $V(m) \wedge U(n)$. # antecedent
    Then $\exists q \in \mathbb{N}, m = 5q + 4$ # first part of antecedent
    Choose $q_1 \in \mathbb{N}, m = 5q_1 + 4$ # instantiate existential
    Then $\exists q \in \mathbb{N}, n = 5q + 3$ # second part of antecedent
    Choose $q_2 \in \mathbb{N}, n = 5q_2 + 3$ # instantiate existential
    Then $m \times n = (5q_1 + 4)(5q_2 + 3) = 5(5q_1q_2 + 3q_1 + 4q_2 + 2) + 2$ # algebra
    Then $\exists q'' \in \mathbb{N}, m \times n = 5q'' + 2$ # $q'' = 5q_1q_2 + 3q_1 + 4q_2 + 2$, and $5, q_1, q_2, 3, 4, 2 \in \mathbb{N}$
    Then $W(m \times n)$ # definition of $W(m \times n)$, since $m, n \in \mathbb{N}$, so $m \times n \in \mathbb{N}$
  Then $(V(m) \wedge U(n)) \Rightarrow W(m \times n)$ # introduced $\Rightarrow$
Then $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, (V(m) \wedge U(n)) \Rightarrow W(m \times n)$ # introduced $\forall$.

3. Use the proof structure of this course to prove the following claim:

$$S_3: \quad \forall m \in \mathbb{N}, (\exists q_1 \in \mathbb{N}, m = 6q_1 + 2) \Rightarrow \big(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\big)$$

Solution:

Assume $m \in \mathbb{N}$. # m is a typical natural number
  Assume $\exists q_1 \in \mathbb{N}, m = 6q_1 + 2$. # antecedent
    Then $m^2 = (6q_1 + 2)^2 = 36q_1^2 + 24q_1 + 4$ # substitute $m = 6q_1 + 2$ into m^2
    Then $m^2 = 6(6q_1^2 + 4q_1) + 4$. # factor previous line
    Then $\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4$. # $q_2 = 6q_1^2 + 4q_1, q_2 \in \mathbb{N}$, since $6, q_1, 4 \in \mathbb{N}$
  Then $(\exists q_1 \in \mathbb{N}, m = 6q_1 + 2) \Rightarrow \big(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\big)$ # introduced $\Rightarrow$
Then $\forall m \in \mathbb{N}, (\exists q_1 \in \mathbb{N}, m = 6q_1 + 2) \Rightarrow \big(\exists q_2 \in \mathbb{N}, m^2 = 6q_2 + 4\big)$. # introduced $\forall$