CSC165 Tutorial #2

Sample Solutions

Winter 2015

Work on these exercises *before* the tutorial. You don't have to come up with complete solutions before the tutorial, but you should be prepared to discuss them with your TA.

1. Consider the statements below:

$$\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$
$$\exists n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$
$$\forall n \in \mathbb{N}, Q(n) \Rightarrow P(n)$$
$$\exists n \in \mathbb{N}, Q(n) \Rightarrow P(n)$$

Evaluate each of the four sentences above using each of the four definitions of predicates P and Q below. Briefly explain your evaluation of each of the sixteen cases.

(a) $P(n) : n < 0 \qquad Q(n) : n^2 < 0$

Solution: $P(n)$ and $Q(n)$ are false for every natural number n, so both the implication and its converse are vacuously true for every $n \in \mathbb{N}$. So, all four sentences evaluate to true for this P and Q.

(b) $P(n) : n < 0 \qquad Q(n) : n^2 > 17$

Solution: The antecedent $P(n)$ is false for every natural number n, so the first two sentences are true for this P and Q. $n = 5$ is a counterexample to the third sentence, so it is false for this P and Q. $n = 2$ makes $Q(n)$ false, and hence is provides an example that the fourth sentence is vacuously true for this P and Q.

(c) $P(n) : n > 3 \qquad Q(n) : n^2 < 0$

Solution: $n^2 \not< 0$ for all natural numbers n, so any implication with $Q(n)$ as the antecedent is vacuously true — the third and four sentences are true for this P and Q. $n = 2$ provides an example where $P(n)$ is false, and hence making the second sentence true. $n = 4$ is a counterexample making the first sentence false.

(d) $P(n) : n > 3 \qquad Q(n) : n^2 > 17$

Solution: $n = 4$ is a counterexample making the first sentence false for this P and Q. $n = 2$ is an example making the implication vacuously true in the second sentence. Since $\sqrt{17} > 4$, the converse is true for every natural number, the third sentence is true for this P and Q, and the particular case $n = 5$ shows the fourth sentence is true for this P and Q (of course, any other natural number would do).

2. Consider the statement:

$\mathbf{S_1}$: For all students, missing an assignment or missing a quiz guarantees not getting a 100% in CSC165.

(a) Write $\mathbf{S_1}$ in logical notation. The logical statement must be **well-formed**. Define all sets and predicate symbols that you use in the logical statement.

Solution: Let S denotes the set of all students, Q denotes the set of all quizzes, A denotes the set of all assignments, $M(x, y)$ denotes x missed y, and F denotes x gets full (100%) mark.

$$\forall x \in S, (\exists a \in A, M(x, a)) \vee (\exists q \in Q, M(x, q)) \Rightarrow \neg F(x).$$

Alternative solution: Let S denotes the set of all students, $MQ(x)$ denotes x missed a quiz, $MA(x)$ denotes x missed an assignment, and F denotes x gets full (100%) mark.

$$\forall x \in S, MA(x) \vee MQ(x) \Rightarrow \neg F(x).$$

(b) Suppose $\mathbf{S_1}$ is true and a student didn't miss any assignments. What, if anything, can be determined about the student missing quizzes and getting 100% in CSC165? Briefly justify your answer.

Solution: Suppose A denotes the student missed an assignment, B denotes the student missed a quiz, and C denotes the student didn't get full mark.

We know that $\mathbf{S_1}$ is true, so we have $A \vee B \Rightarrow C$.

We also know that A is false. So, three cases are possible: (1) B and C are false. (2) B is false and C is true. (1) B and C are true.

So knowing that A is false tells us nothing.

(c) Suppose $\mathbf{S_1}$ is true and a student got 100% in CSC165. What, if anything, can be determined about the student missing assignments and quizzes? Briefly justify your answer.

Solution: We know that C is false. So to keep $\mathbf{S_1}$ true, the antecedent of the implication must be false, which means that A and B are false.

3. Translate the following sentences into logical notation. The logical sentences must be **well-formed**. Define all sets and predicate symbols that you use in the logical sentences.

(a) If some NP-complete problem can be solved efficiently, then every NP-complete problem can be solved efficiently.

Solution: Let NPC denotes the set of all NP-complete problems and $SE(x)$ denotes that x can be solved efficiently.

$$(\exists x \in NPC, SE(x)) \Rightarrow (\forall x \in NPC, SE(x)).$$

(b) Some courses have exactly one prerequisite course.

Solution: Let C denotes the set of all courses and $P(y, x)$ denotes that y is a prerequisite for x.

$$\exists x \in C, \exists y \in C, P(y, x) \wedge (\forall z \in C, P(z, x) \Rightarrow (z = y)).$$

(c) Some courses have the same prerequisite courses.

$$\exists x \in C, \exists y \in C, \forall z \in C, P(z, x) \Leftrightarrow P(z, y)$$

(d) For every person, being a student is necessary for enrollment.

Solution: Let P denotes the set of all persons, $E(x)$ denotes that x is enrolled and $S(x)$ denotes x is a student.

$$\forall x \in P, E(x) \Rightarrow S(x).$$

4. Give the contrapositive and converse of Question 3(d) in English.

Contrapositive: For every person, if the person is not a student, then s/he is not enrolled.

Converse: For every person, if the person is a student, then s/he is enrolled.

Alternative #1: For every person, being enrolled is necessary for being a student.

Alternative #2: For every person, being a student is sufficient for enrollment.