PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

Term Test #2

CSC 165H1
Section L0101
Duration — 60 minutes

No aids allowed

PLEASE HAND IN

Last Name: ______________________________

First Name: ______________________________

Student No: ______________________________

Do **not** *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 3 questions on 5 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Answer each question directly on the exam paper, in the space provided, and use the reverse side of the previous page for rough work. If you need more space for one of your solutions, use a blank page, **indicate** clearly the part of your work that should be marked, and indicate the **number of the page** where your answer is on the page where the question appears.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do– part marks will be given for showing that you know the general structure of an answer, even if your solutions is incomplete.

Good Luck!

IMPORTANT: For all questions, you **must** use the proof structures and format of this course. Otherwise, you won't get full mark even if your answer is correct.

Question 1. [13 MARKS]

Use the proof structure from this course to prove or disprove the following claims.

Note: You will receive **part marks** just for correctly writing the proof structure (aka the proof outline).

Part (a) [6 MARKS] $\mathbf{S_1}$: For all integers n, $7n+3$ is divisible by 7.

Solution: The claim is false, so I disprove it.

Here's the translation of the claim:

$$\forall n \in \mathbb{Z}, 7 \mid (7n+3).$$

To disprove it, I must prove the negation of the claim:

$$\exists n \in \mathbb{Z}, 7 \nmid (7n+3).$$

Let $n = 0$. Then $n \in \mathbb{Z}$. # since $0 \in \mathbb{Z}$
Then $7n+3 = 7 \times 0 + 3 = 3$. # substitute n by 0
Then $7 \nmid 7n+3$. # 7 does not divide 3
Then $\exists n \in \mathbb{Z}, 7 \nmid (7n+3)$. # introduce $\exists$

Part (b) [7 MARKS] $\mathbf{S_2}$: For all integers if n is even, then $7n+4$ is even.

Solution: The claim is true.

Here's the translation of the claim:

$$\forall n \in \mathbb{Z}, Even(n) \Rightarrow Even(7n+4).$$

Assume $n \in \mathbb{Z}$. # n is a typical integer
  Assume n is even. # antecedent
    Then exists $k_0 \in \mathbb{Z}$ such that $n = 2k_0$. # definition of even numbers
    Then $7n+4 = 14k_0 + 4$. # substitute n by $2k_0$ and algebra
    Then exists $k_1 \in \mathbb{Z}$ such that $7n+4 = 2k_1$. # $k_1 = 7k_0 + 2$, and $k_1 \in \mathbb{Z}$
    Then $7n+4$ is even. # definition of even numbers
  Then $Even(n) \Rightarrow Even(7n+4)$. # contrapositive is equivalent to the implication
Then $\forall n \in \mathbb{Z}, Even(n) \Rightarrow Even(7n+4)$. # introduce $\forall$

Question 2. [12 MARKS]

Use the proof structure from this course to prove $\mathbf{S_3}$.

Note: You will receive **part marks** just for correctly writing the proof structure (aka the proof outline).

$\mathbf{S_3}$: For all integers n, if $n^2 + 5$ is odd, then n is even.

Solution: Here's the translation of the claim:

$$\forall n \in \mathbb{Z}, Odd(n^2 + 5) \Rightarrow Even(n).$$

Assume $n \in \mathbb{Z}$. # n is a typical integer
  Assume n is odd. # antecedent of the contrapositive
    Then exists $k_0 \in \mathbb{Z}$ such that $n = 2k_0 + 1$. # definition of odd numbers
    Then $n^2 + 5 = (2k_0 + 1)(2k_0 + 1) + 5 = 4k_0^2 + 4k_0 + 6$. # substitute n by $2k_0 + 1$ and algebra
    Then exists $k_1 \in \mathbb{Z}$ such that $n^2 + 5 = 2k_1$. # $k_1 = 2k_0^2 + 2k_0 + 3$, and $k_1 \in \mathbb{Z}$
    Then $n^2 + 5$ is even. # definition of even numbers
  Then $Odd(n) \Rightarrow Even(n^2 + 5)$. # introduce $\Rightarrow$
  Then $Odd(n^2 + 5) \Rightarrow Even(n)$. # contrapositive is equivalent to the implication
Then $\forall n \in \mathbb{Z}, Odd(n^2 + 5) \Rightarrow Even(n)$. # introduce $\forall$

Question 3. [15 MARKS]

Use **proof by contradiction** to prove that there is no integer n such that ($n \equiv 5$ mod 6) and ($n \equiv 3$ mod 12).

Note 1: You will receive **part marks** just for correctly writing the proof structure (aka the proof outline).

Note 2: You **must** use the proof structure from this course.

Hint: Recall that for integers x, y, z, the notation $x \equiv y \bmod z$ means "$x - y$ is a multiple of z."

Solution: Here's the translation of the claim:

$$\neg(\exists n \in \mathbb{Z}, (n \equiv 5 \bmod 6) \wedge (n \equiv 3 \bmod 12)).$$

We must assume the negation of the claim and then derive a contradiction.

Assume $\exists n \in \mathbb{Z}, (n \equiv 5 \bmod 6) \wedge (n \equiv 3 \bmod 12)$. # to derive a contradiction
  Then exists $k_0 \in \mathbb{Z}$ such that $n - 5 = 6k_0$. # definition *mod*
  Then $n = 6k_0 + 5$. # add 5 to both sides of the above equality
  Also exists $k_1 \in \mathbb{Z}$ such that $n - 3 = 12k_1$. # definition *mod*
  Then $n = 12k_1 + 3$. # add 3 to both sides of the above equality
  Then $12k_1 + 3 = 6k_0 + 5$. # both are equal to n
  Then $12k_1 - 6k_0 = 2$. # subtract $6k_0 + 3$ from both sides
  Then $2k_1 - k_0 = 2/6$. # divide both sides by 6
  Contradiction! # $2k_1 - k_0 \in \mathbb{Z}$, but 2/6 is not an integer
Then $\neg(\exists n \in \mathbb{Z}, (n \equiv 5 \bmod 6) \wedge (n \equiv 3 \bmod 12))$. # assuming the negation leads to a contradiction

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1: ________/13

2: ________/12

3: ________/15

TOTAL: ________/40