CSC165, Winter 2015
Solutions for Lecture Exercises
Feb 13, 2015

1. Consider the definition of the floor function:

$$\mathbf{D_1} : \forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, (y = \lfloor x \rfloor) \Leftrightarrow (y \leq x) \wedge (\forall z \in \mathbb{Z}, (z \leq x) \Rightarrow (z \leq y)).$$

   Use $\mathbf{D_1}$ to prove $\forall x \in \mathbb{R}, (\lfloor x \rfloor > x - 1)$.

   Proof:

   Assume $x \in \mathbb{R}$. # x is a typical element of $\mathbb{R}$
   Then $\lfloor x \rfloor, \lfloor x \rfloor + 1 \in \mathbb{Z}$. # by definition of the floor function and $\mathbb{Z}$ is closed under +
   And $\lfloor x \rfloor + 1 > \lfloor x \rfloor$. # add $\lfloor x \rfloor$ to $1 > 0$
   Then $\lfloor x \rfloor + 1 > x$. # by contrapos. of the implication in $\mathbf{D_1}$ which is $\forall z \in \mathbb{Z}, (z > \lfloor x \rfloor) \Rightarrow (z > x)$
   Then $\lfloor x \rfloor > x - 1$. # subtract 1 from both sides
   Then $\forall x \in \mathbb{R}, (\lfloor x \rfloor > x - 1)$. # introduced $\forall$

2. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor$.

   Solution: To derive a contradiction, we assume the negation of the claim.

   Proof by contradiction:

   Assume $\neg(\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor)$ # to derive contradiction

   Then $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, (x > y) \wedge (\lfloor x \rfloor < \lfloor y \rfloor)$ # the negation
   Let $x_0 \in \mathbb{R}, y_0 \in \mathbb{R}$ be such that $(x_0 > y_0) \wedge (\lfloor x_0 \rfloor < \lfloor y_0 \rfloor)$. # instantiate $\exists$
   Then $\lfloor x_0 \rfloor < \lfloor y_0 \rfloor$ # conjunction elimination
   And $\lfloor x_0 \rfloor \in \mathbb{Z}, \lfloor y_0 \rfloor \in \mathbb{Z}$ # by definition of floor
   Then $\lfloor x_0 \rfloor + 1 \leq \lfloor y_0 \rfloor$ # the smallest possible difference between two distinct integers is 1
   Then $\lfloor x_0 \rfloor + 1 \leq y_0$ # since $\lfloor y_0 \rfloor \leq y_0$ by definition of $\lfloor y_0 \rfloor$
   Then $\lfloor x_0 \rfloor + 1 < x_0$ # $y_0 < x_0$ by the assumption and $<$ is transitive
   And $\lfloor x_0 \rfloor + 1 \in \mathbb{Z}$ # $1, \lfloor x_0 \rfloor \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under +
   Then $\lfloor x_0 \rfloor + 1 \leq \lfloor x_0 \rfloor$ # by definition of $\lfloor x_0 \rfloor$ that $\forall z \in \mathbb{Z}, z \leq x_0 \Rightarrow z \leq \lfloor x_0 \rfloor$
   Then $1 \leq 0$ # subtract $\lfloor x_0 \rfloor$ from both sides, and contradiction with that $1 > 0$

   Then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor$ # negation of assumption because of contradiction

3. For $x \in \mathbb{R}$, define $|x|$ by

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Prove that $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

Solution: For each variable we must consider two cases: in one case the variable is ≥ 0 and in the other one the variable is < 0. Since we have two variables we must have four cases in our proof.

Proof:

Assume $x, y \in \mathbb{R}$. # x and y are typical elements of $\mathbb{R}$
  Case 1: $x < 0$ and $y < 0$.
    Then $|x| = -x$ and $|y| = -y$. # definition of $|x|$ and $|y|$
    Then $|x||y| = (-x)(-y) = xy$. # since $(-1)^2 = 1$
    And $xy > 0$. # the product of two negative numbers is positive
    Then $xy = |xy|$. # definition of $|xy|$ when $xy \geq 0$ Then $|x||y| = |xy|$.
  Case 2: $x < 0$ and $y \geq 0$.
    Then $|x| = -x$ and $|y| = y$. # definition of $|x|$ and $|y|$
    Then $|x||y| = -xy$. # algebra
    And $xy \leq 0$. # product of a negative and a non-negative number is either 0 or negative
    Case 2.1: Assume $xy < 0$.
      Then $|xy| = -xy$. # by the definition of $|xy|$
    Case 2.2: Assume $xy = 0$.
      Then $|xy| = 0 = -xy$. # product of any number and 0 is 0, and by the definition of $|xy|$
    Then $|xy| = -xy$. # true for both possible cases
    Then $|x||y| = |xy|$. # we showed that both are equal to $-xy$
  Case 3: $x \geq 0$ and $y < 0$.
    Then $|x| = x$ and $|y| = -y$. # definition of $|x|$ and $|y|$
    Then $|x||y| = -xy$. # algebra
    And $xy \leq 0$. # product of a non-negative number with a negative number is non-positive
    Case 3.1: Assume $xy < 0$.
      Then $|xy| = -xy$. # by the definition of $|xy|$
    Case 3.2: Assume $xy = 0$.
      Then $|xy| = 0 = -xy$. # product of any number and 0 is 0, and by the definition of $|xy|$
    Then $|xy| = -xy$. # true for both possible cases
    Then $|x||y| = |xy|$. # we showed that both are equal to $-xy$
  Case 4: $x \geq 0$ and $y \geq 0$.
    Then $|x||y| = xy$. # $|x| = x$ and $|y| = y$ by definition of $|x|$ and $|y|$
    And $xy \geq 0$. # the product of two non-negative numbers is non-negative
    Then $|xy| = xy$. # definition of $|xy|$
    Then $|x||y| = |xy|$. # both are equal to xy
  Then $|x||y| = |xy|$. # true for all possible cases
Then $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $|x||y| = |xy|$. # introduced $\forall$