CSC165: Final Exam Review

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Announcements

- Final exam: April 9, 9-12, EX200
- Support:
  - Office hours: (over 40 hours) check course website for update
  - Help Centre schedule: link [1]

[1] The exam times listed in the Help Centre calendar are unofficial and may be incorrect. It is very important that you double-check your exam times on the official time table.

Overview

- Logical notation: translate sentences using logical expression
- Evaluate statement: true or false
  - how to evaluate: truth table or manipulation rule to prove it
  - some concepts: tautology / satisfiability / unsatisfiability
- Formal proof: statement true or false Prove a statement using logical consequence, to derive a conclusion from given assumptions
  - direct proof: $P \Rightarrow Q$
  - indirect proof: contraposition & contradiction
- Algorithm analysis using asymptotic notation
  - $\mathcal{O}, \Omega, \Theta$
  - proof

Logical notation: use logical expression

- Translate sentences using logical expression
  - quantifiers: $\forall, \exists$
  - predicates: $P(x)$
  - $\wedge, \vee, \neg$
  - $\rightarrow, \longleftrightarrow$
- Evaluate statement: true or false
  - how to evaluate: truth table / manipulation rules to prove it
  - some concepts: tautology / satisfiability / unsatisfiability

Review exercises

- test 1
- assignment 1
- tutorial 1-3

Manipulation Rules

- Identity
  $P \wedge (Q \vee \neg Q) \iff P$
  $P \vee (Q \wedge \neg Q) \iff P$
- Idempotency
  $P \wedge P \iff P$
  $P \vee P \iff P$
- Commutativity
  $P \wedge Q \iff Q \wedge P$
  $P \vee Q \iff Q \vee P$
  $(P \Leftrightarrow Q) \iff (Q \Leftrightarrow P)$
- Associativity
  $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$
  $(P \vee Q) \vee R \iff P \vee (Q \vee R)$
- Distributivity
  $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- DeMorgan's laws
  $\neg(P \wedge Q) \iff \neg P \vee \neg Q$
  $\neg(P \vee Q) \iff \neg P \wedge \neg Q$

Formal proofs: prove statements true or false

Prove a statement true of false, using derived logical consequences from given assumptions (P), to a conclusion (Q).

- Direct proof: $P \Rightarrow Q$
- Indirect proof:
  - by contraposition: $\neg Q \Rightarrow \neg P$
  - by contradiction: $\neg(P \Rightarrow Q)$

Formal proofs

Reviewing exercises

- test 2
- assignment 2
- tutorial 4-5

TIPS

- Proof techniques:
  direct / indirect proofs / disproofs (negate and prove the negation)
- Proof structures:
  - Keywords:
    assume, pick, let, then – should match the quantifier (ordered) in the statement to prove.
  - indentations
  - comment/justification: critical for important steps.
- Types of questions: related to what we have practiced in the course

Proof Structure

General Structure of a Typical Proof

- Given a set of ASSUMPTIONS, prove a CLAIM.
  - Start from the assumptions.
  - Derive a logical consequence, based on the assumptions.
  - Add the new consequence to the original set of assumptions.
  - Continue until the claim can be derived from the assumptions.

Prove $P \Rightarrow Q$

- Given P, prove Q:

Assume $\mathbf{P}$. # Given assumption

Then $\mathbf{R_1}$. # by P or another known fact

Then $\mathbf{R_2}$. # by R_1 or another known fact

$\vdots$

Then $\mathbf{R_n}$. # by R_{n-1} or another known fact

Then $\mathbf{Q}$. # by R_n or another known fact

Logical proofs: exercises

Algorithm analysis using asymptotic notation

- $\mathcal{O}, \Omega, \Theta$
- find & prove bounds
  - Worst-case run time analysis: exact step expression
  - polynomial expressions
  - limit definition
  - induction

Definitions

$$\mathcal{O}(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \leqslant cf(n)\}.$$

$$\Omega(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \geqslant cf(n)\}.$$

$$\Theta(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow c_1 f(n) \leqslant g(n$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow \frac{f(n)}{g(n)} > \varepsilon$$

Induction

Basic steps for simple induction:

prove the base case (which may now be greater than 0)
prove the induction step

Asymptotic notation: exercises

Reviewing exercises

- assignment 3
- tutorial 6-7

TIPS

- for polynomials:
  use the standard procedure with a chain of overestimate and underestimate
- for non-polynomials:
  use calculus (possibly L'Hopital) to prove limit, translate to definition of limit, then relate it to the definition of $\mathcal{O}$
  review calculus: know basic rules of taking derivatives

Asymptotic notation: exercises

1. $n^6 \notin \mathcal{O}(3n^5)$.
2. $n^2 + n \in \Omega(15n^2 + 3)$.
   $n^2 + n \in \mathcal{O}(15n^2 + 3)$.
3. $3n^2 + 2n \in \Theta(n^2)$.
4. $n * floor(n/2) \in \mathcal{O}(n^2)$.
5. $f(n) = \max(n^2, 100)(3n + 1) - 5$, find the tight bound for $f(n)$, and prove it.

Asymptotic notation: exercises

1. Prove: $n^6 \notin \mathcal{O}(3n^5)$

Let $g(n) = n^6, f(n) = 3n^5$

We rely on the fact that $\lim_{n\to\infty} n^6/3n^5 = \infty$.[2]

Assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$. # arbitrary values

Then $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow n^6/3n^5 > c$.

definition of $\lim_{n\to\infty} n^6/3n^5 = \infty$, then $n^6/3n^5 > \varepsilon$, let $\varepsilon = c$

Let $n' = \max(B, n_0)$, $n' \in \mathbb{N}$

Then $n' \geqslant n_0, n' \geqslant B, n'^6 > c \cdot 3n'^5$

max definition; limit definition: $n^6/3n^5 > c$

Then $n' \geqslant B \wedge (n'^6 > c \cdot 3n'^5)$. # introduce $\wedge$

Then $\exists n \in \mathbb{N}, n \geqslant B \wedge g(n) > cf(n)$. # introduce $\exists$

Then $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geqslant B \wedge g(n) > cf(n)$. # introduce $\forall$

[2] Applying l'Hôpital's Rule

Let $f(n) = \max(n^2, 100)(3n+1) - 5$, and $g(n) = n^3$. Then $f \in \Theta(g)$ - 1

SOLUTION:

Let $c_1 = 1$ and $c_2 = 395$; then $c_1, c_2 \in \mathbb{R}^+$

Let $B' = 1$; then $B' \in \mathbb{N}$

Assume $n \geq B', n \in \mathbb{N}$

Case 1: $n \leq 10$, then $\max(n^2, 100) = 100$

// Proving $f \geq c \cdot g$

$$\begin{aligned}\text{Then } f(n) = 300n + 100 - 5 &= 300n + 95 \\ &\geq 300n \quad \# \, 95 \in \mathbb{N} \\ &\geq (n^2)n \quad \# \, n^2 \leq 100 < 300, n \leq 10 \\ &= 1 \cdot n^3 = c_1 g(n)\end{aligned}$$

Then $f(n) \geq c_1 \cdot g(n)$

// Proving $f \leq c \cdot g$

$$\begin{aligned}\text{Then } f(n) = 300n + 100 - 5 &= 300n + 95 \\ &\leq 300n + 95n \quad \# \, n \in \mathbb{N}, n \geq 1 \\ &\leq 395n^3 \quad \# \, n \in \mathbb{N}, n^3 > n \\ &= 395 \cdot n^3 = c_2 g(n)\end{aligned}$$

Then $f(n) \leq c_2 \cdot g(n)$

Let $f(n) = \max(n^2, 100)(3n+1) - 5$, and $g(n) = n^3$. Then $f \in \Theta(g)$ - 2

SOLUTION:

...

Case 2: $n > 10$, then $\max(n^2, 100) = n^2$;
// Proving $f \geq c \cdot g$

$$
\begin{aligned}
\text{Then } f(n) &= 3n^3 + n^2 - 5 \\
&\geq 3n^3 - 5 \quad \# \, n^2 \in \mathbb{N} \\
&= n^3 + 2n^3 - 5 \\
&\geq 1 \cdot n^3 = c_1 g(n) \quad \# \text{ for } n > 10, 2n^3 - 5 > 0
\end{aligned}
$$

Then $f(n) \geq c_1 \cdot g(n)$
// Proving $f \leq g$

$$
\begin{aligned}
\text{Then } f(n) &= 3n^3 + n^2 - 5 \\
&\leq 3n^3 + n^2 \quad \# \, 5 \in \mathbb{N} \\
&\leq 3n^3 + n^3 = 4n^3 \quad \# \, n \in \mathbb{N} \\
&\leq 395 \cdot n^3 = c_2 g(n)
\end{aligned}
$$

Then $f(n) \leq c_2 \cdot g(n)$

Let $f(n) = \max(n^2, 100)(3n + 1) - 5$, and $g(n) = n^3$. Then $f \in \Theta(g)$ - 3

SOLUTION:

Let $c_1 = 1$ and $c_2 = 395$; then $c_1, c_2 \in \mathbb{R}^+$

Let $B' = 1$; then $B' \in \mathbb{N}$

Assume $n \geq B', n \in \mathbb{N}$

Case 1: $n \leq 10$, then $\max(n^2, 100) = 100$

...

Case 2: $n > 10$, then $\max(n^2, 100) = n^2$;

...

Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$

Then $\exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$

Then $\exists c_1, c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$

Then $f \in \Theta(g)$