Chapter 4

Algorithm Analysis and Asymptotic Notation

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Announcements

- **Assignment 2** is due next Friday Mar 06, before midnight.
- **TA office Hours** for Assignment 2:
  - Tuesday, Mar 03, 4-6pm in **BA3201**
  - Thursday, Mar 05, 10am-noon, 4-8pm in **BA3201**
- **Term Test 2**:
  - Section **L0101**: Tuesday **Mar 10, 2:10-3:30** Location: **EX100**
  - Section **L0201**: Thursday **Mar 12, 2:10-3:30** Location: **EX100**
  - You **must** write the test in the section that you are enrolled in, unless you have talked to the instructors and they allowed you to switch.
  - **Content:** **Chapter 3**. Review **lecture** and **course** notes!

Today's Topics

- **Review: Asymptotic Notation**
- **Example: Asymptotic Notation**
- **Limits and Asymptotic Notation**

Chapter 4

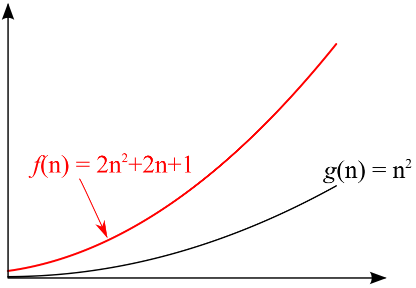
Algorithm Analysis and Asymptotic Notation

Review: Asymptotic Notation

Review: Asymptotic Notation

Big-Oh

- $\mathbf{f} \in \mathcal{O}(\mathbf{g})$: g is an **upper bound** of f.
  - For sufficiently large values of n, $g(n)$ multiply by a constant is always **greater than** $f(n)$.
- $\exists \mathbf{c} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow f(n) \leq \mathbf{c}.g(n)$



Review: Asymptotic Notation

Big-Oh

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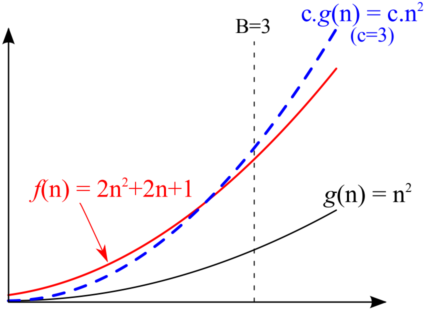
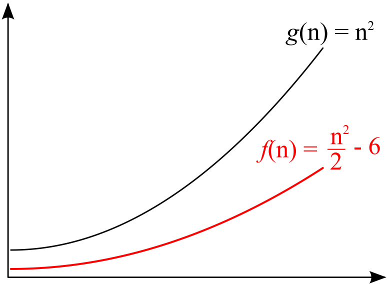


$$(\mathbf{2n^2 + 2n + 1}) \in \mathcal{O}(\mathbf{n^2})$$

Review: Asymptotic Notation

Big-Omega

- $\mathbf{f} \in \boldsymbol{\Omega}(\mathbf{g})$: g is a **lower bound** of f.
  - For sufficiently large values of n, $g(n)$ multiply by a constant is always **less than** $f(n)$.
- $\exists \mathbf{c} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow f(n) \geq \mathbf{c}.g(n)$



Review: Asymptotic Notation

Big-Omega

- $\mathbf{f} \in \mathbf{\Omega}(\mathbf{g})$: g is a **lower bound** of f.
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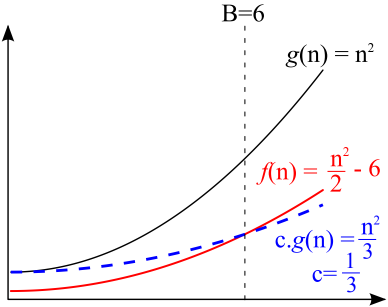
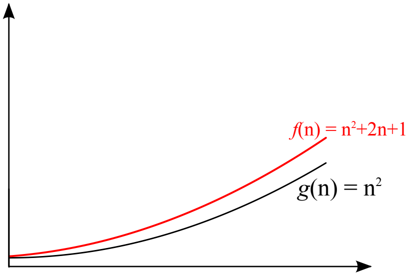


$$(\frac{\mathbf{n^2}}{\mathbf{2}} - \mathbf{6}) \in \mathbf{\Omega}(\mathbf{n^2})$$

Review: Asymptotic Notation

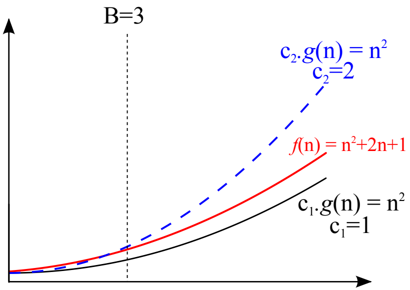
Big-Theta

- $\mathbf{f} \in \mathbf{\Theta(g)}$: g is a **tight bound** of f.
  - For sufficiently large values of n, $g(n)$ is both an **upper bound** and a **lower bound** for $f(n)$.
- $\exists \mathbf{c_1} \in \mathbb{R}^+, \exists \mathbf{c_2} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow \mathbf{c_1}.g(n) \leq f(n) \leq \mathbf{c_2}.g(n)$



Big-Theta

- $\mathbf{f} \in \mathbf{\Theta}(\mathbf{g})$: g is a **tight bound** of f.
  - For sufficiently large values of n, $g(n)$ is both an **upper bound** and a **lower bound** for $f(n)$.
- $\exists \mathbf{c_1} \in \mathbb{R}^+, \exists \mathbf{c_2} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow \mathbf{c_1}.g(n) \leq f(n) \leq \mathbf{c_2}.g(n)$



$(\mathbf{n^2} + \mathbf{2n} + \mathbf{1}) \in \mathbf{\Theta}(\mathbf{n^2})$

Chapter 4

Algorithm Analysis and Asymptotic Notation

Example: Asymptotic Notation

Example: Proving Lower Bound

- Let $g(n) = n^2$, $f(n) = \frac{n^2+n-4}{3}$.
  Prove that $\mathbf{f} \in \mathbf{\Omega}(\mathbf{g})$
- $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \frac{n^2+n-4}{3} \geq c.n^2$

Proof Structure

Let $c = ...$, $B = ...$ Then $c \in \mathbb{R}^+$, $B \in \mathbb{N}$.
Assume $n \in \mathbb{N}$. # n is a typical natural number
    Assume $n \geq B$. # antecedent
        .
        .
        .
    Then $n \geq B \implies \frac{n^2+n-4}{3} \geq c.n^2$. # introduce $\implies$
Then $\forall n \in \mathbb{N}, n \geq B \implies \frac{n^2+n-4}{3} \geq c.n^2$. # introduce $\forall$
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies \frac{n^2+n-4}{3} \geq c.n^2$. # introduce $\exists$

Example: Proving Lower Bound

- Let $g(n) = n^2$, $f(n) = \frac{n^2+n-4}{3}$.
  Prove that $\mathbf{f} \in \mathbf{\Omega(g)}$
- $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq c.g(n)$

Scratch Work

- Find an appropriate value for **c** and **B**.
- First choose a value for **c**: $\mathbf{\frac{1}{4}}$ seems reasonable.
- Assuming $\mathbf{c = \frac{1}{4}}$, find an appropriate value for **B**:

$$\frac{n^2}{4} \leq \frac{n^2+n-4}{3} \iff 3n^2 \leq 4n^2 + 4n - 16 \quad \text{\# multiply both sides by 12}$$
$$\iff 0 \leq n^2 + 4n - 16 \quad \text{\# subtract } 3n^2 \text{ from both sides}$$
$$\iff 2 < n \quad \text{\# solving the quadratic inequality}$$

- $\mathbf{B = 3}$

Example: Proving Lower Bound

- Let $g(n) = n^2$, $f(n) = \frac{n^2+n-4}{3}$.
  Prove that $\mathbf{f \in \Omega(g)}$
- $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \frac{n^2+n-4}{3} \geq c.n^2$

Proof Structure

Let $\mathbf{c = \frac{1}{4}}$, $\mathbf{B = 3}$. Then $c \in \mathbb{R}^+$, $B \in \mathbb{N}$.
Assume $n \in \mathbb{N}$. # n is a typical natural number
  Assume $n \geq B$. # antecedent
    $3n^2 \geq 3n^2$. # $3n^2 = 3n^2$
    $4n^2 + 4n - 16 \geq 3n^2$. # $n \geq 3$, so $n^2 + 4n - 16 \geq 5$
    $\frac{n^2+n-4}{3} \geq \frac{n^2}{4}$. # divide both sides by 12
    $\frac{n^2+n-4}{3} \geq c.n^2$. # $\mathbf{c = \frac{1}{4}}$
  Then $n \geq B \implies \frac{n^2+n-4}{3} \geq c.n^2$. # introduce $\implies$
Then $\forall n \in \mathbb{N}, n \geq B \implies \frac{n^2+n-4}{3} \geq c.n^2$. # introduce $\forall$
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies \frac{n^2+n-4}{3} \geq c.n^2$. # introduce $\exists$

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Algorithm Analysis and Asymptotic Notation

Limits and Asymptotic Notation

Limits and Asymptotic Notation

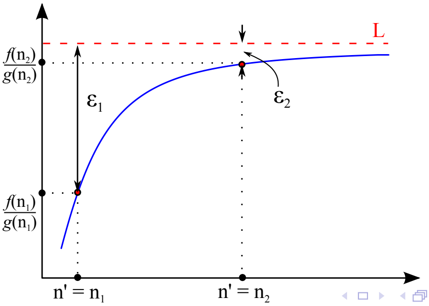
- Suppose $\mathbf{f} \in \mathcal{O}(\mathbf{g})$.
  Then $\mathbf{f(n)} \leq \mathbf{c.g(n)}$, $c \in \mathbb{R}^{\geq 0}$, n is sufficiently large.
- Then $\frac{\mathbf{f(n)}}{\mathbf{g(n)}} \leq \mathbf{c}$ (assuming that $g(n) \neq 0$) for **sufficiently large** values of n.
- $\lim_{\mathbf{n} \to \infty} \frac{\mathbf{f(n)}}{\mathbf{g(n)}} \leq \mathbf{c}$.

We should be able to use Limits in proving/disproving function bounds!!

Limits and Asymptotic Notation

Reminder: Limits

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$
- $\forall n \in \mathbb{N}, n \geq n_1 \Rightarrow \mid \frac{f(n)}{g(n)} - L \mid < \varepsilon_1.$
- $\forall n \in \mathbb{N}, n \geq n_2 \Rightarrow \mid \frac{f(n)}{g(n)} - L \mid < \varepsilon_2.$
- $\forall \varepsilon \in \mathbb{R}^+, \exists \mathbf{n}' \in \mathbb{N}, \forall \mathbf{n} \in \mathbb{N}, \mathbf{n} \geq \mathbf{n}' \Rightarrow \mid \frac{\mathbf{f(n)}}{\mathbf{g(n)}} - \mathbf{L} \mid < \varepsilon.$



Limits and Asymptotic Notation

- $\forall \varepsilon \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \mid \frac{f(n)}{g(n)} - L \mid < \varepsilon.$
  is equivalent to
  $$\forall \varepsilon \in \mathbb{R}^+, \exists \mathbf{n}' \in \mathbb{N}, \forall \mathbf{n} \in \mathbb{N}, \mathbf{n} \geq \mathbf{n}' \Rightarrow -\varepsilon < \frac{\mathbf{f(n)}}{\mathbf{g(n)}} - \mathbf{L} < \varepsilon.$$
  is equivalent to
  $$\forall \varepsilon \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon.$$

Reminder: Limits

- $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$
  $\Longleftrightarrow$
  $\forall \varepsilon \in \mathbb{R}^+, \exists \mathbf{n}' \in \mathbb{N}, \forall \mathbf{n} \in \mathbb{N}, \mathbf{n} \geq \mathbf{n}' \Rightarrow \mathbf{L} - \varepsilon < \frac{\mathbf{f(n)}}{\mathbf{g(n)}} < \mathbf{L} + \varepsilon$

Proving Upper Bound using Limits

Reminder: Limits

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$
  $\iff$
  $\forall \varepsilon \in \mathbb{R}^+, \exists \mathbf{n}' \in \mathbb{N}, \forall \mathbf{n} \in \mathbb{N}, \mathbf{n} \geq \mathbf{n}' \Rightarrow \mathbf{L} - \varepsilon < \frac{\mathbf{f(n)}}{\mathbf{g(n)}} < \mathbf{L} + \varepsilon$

Assume $\lim_{n\to\infty} f(n)/g(n) = L$.

Then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies L - 1 < \frac{f(n)}{g(n)} < L + 1$. # definition of limit for $\varepsilon = 1$

Then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies f(n) \leq (L+1)g(n)$.

Then $f \in \mathcal{O}(g)$. # definition of $\mathcal{O}$, with $B = n'$ and $c = L + 1$

Hence, $\lim_{n\to\infty} f(n)/g(n) = L \implies f \in \mathcal{O}(g)$.

Disproving Upper Bound using Limits

Disproving Upper Bound

- Let $f(n) = 2^n$ and $g(n) = n$. Show that $f \notin \mathcal{O}(g)$.
- Prove the **negation** of the following

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c.g(n)$$

- Prove

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > c.g(n)$$

- Note: $\lim_{n \to \infty} \frac{2^n}{n} = \infty$.

Disproving Upper Bound using Limits

Reminder: Special Case of Limits

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$
  $\Longleftrightarrow$
  $\forall \varepsilon \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies \frac{f(n)}{g(n)} > \varepsilon$

- $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c.n$

Assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$. # arbitrary values

Then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies 2^n/n > c$. # definition of $\lim_{n\to\infty} 2^n/n = \infty$ with $\varepsilon = c$

Let n_1 be such that $\forall n \in \mathbb{N}, n \geq n_1 \implies 2^n/n > c$. # instantiate n'

Let $n_0 = \max(B, n_1)$. Then $n_0 \in \mathbb{N}$.

Then $n_0 \geq B$. # by definition of max

Then $2^{n_0} > cn_0$. # by the assumption above $2^{n_0}/n_0 > c$, since $n_0 \geq n_1$

Then $n_0 \geq B \wedge 2^{n_0} \geq c.n_0$. # introduce $\wedge$

Then $\exists n \in \mathbb{N}, n \geq B \wedge 2^n \geq c.n$ # introduce $\exists$

Then $\forall c \in \mathbb{R}, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > c.n$ # introduce $\forall$

Take Home Problem

Use Limits in proving/disproving lower bound