CSC165 Mathematical Expression and Reasoning for Computer Science

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Announcements

TERM TEST 1:

- Paper pick-up Time: Wed. office hours 3-5
- Request for remark deadline: Feb. 25(L0201) / Feb 27(L0101).
- Written request only: form will be available on course website this week.

Exercise 1:

PROVE OR DISPROVE: $\forall x, y \in \mathbb{R}, x < y \Rightarrow (\exists z \in \mathbb{R}, x < z \wedge z < y)$.

Solution: The statement is true. It says that if you have one real number less than another, then there is another real number strictly between them. The way to prove this is to construct z in terms of x and y. One way to do this is to make z the midpoint.

Assume x, y are real numbers. # in order to introduce $\forall$

> Assume $x < y$. # in order to introduce $\Rightarrow$
>
> > Pick $z = (x + y)/2.0$. Then $z \in \mathbb{R}$. # $\mathbb{R}$ closed under $+, /2.0$
> >
> > Then $z < y$. # Since
> > $x < y \Rightarrow x + y < y + y \Rightarrow (x + y)/2.0 < (y + y)/2.0$.
> >
> > Also $x < z$. # Since
> > $x < y \Rightarrow x + x < x + y \Rightarrow (x + x)/2.0 < (x + y)/x.0$
> >
> > So $x < z \wedge z < y$. # introduced conjunction.
> >
> > So $\exists z \in \mathbb{R}, x < z \wedge z < y$. # introduced $\exists$
>
> Then $x < y \Rightarrow (\exists z \in \mathbb{R}, x < z \wedge z < y)$. # introduced $\Rightarrow$

Then $\forall x, y \in \mathbb{R}, x < y \Rightarrow (\exists z \in \mathbb{R}, x < z \wedge z < y)$. # introduced $\forall$

Exercise 2:

PROVE OR DISPROVE: $\forall m, n \in \mathbb{N}, m < n \Rightarrow (\exists k \in \mathbb{N}, m < k \wedge k < n)$

Solution: The claim is false. To disprove it, just find a pair of consecutive natural numbers, and observe that there is no natural number between them (since they increment by 1). I prove the negation of the original statement.

$$\exists m, n \in \mathbb{N}, m < n \wedge (\forall k \in \mathbb{N}, m \geq k \vee k \geq n)$$

Pick $m = 1, n = 2$. Then $m, n \in \mathbb{N}$. # 1, 2 are natural numbers

Then $m < n$. # $1 < 2$.

Assume $k \in \mathbb{N}$. # in order to introduce $\forall$

Assume $k > m = 1$. # in order to introduce $\Rightarrow$

Then $k \geq 2$. # Successor to 1 is $1 + 1 = 2$ in $\mathbb{N}$

Then $k > m \Rightarrow k \geq 2 = n$. # introduced $\forall$

Then $k \leq m \vee k \geq n$. # equivalent to $k > m \Rightarrow k \geq n$.

Then $\forall k \in \mathbb{N}, k \leq m \vee k \geq n$. # introduced $\forall$

Then $m < n \wedge (\forall k \in \mathbb{N}, m \geq k \vee k \geq n)$. # introduced conjunction

Exercise 3-1:

PROVE OR DISPROVE: For all quadruples of positive real numbers w, x, y, z, If $w/x < y/z$ then:

$$\left(\frac{w}{x} < \frac{w+y}{x+z}\right) \wedge \left(\frac{w+y}{x+z} < \frac{y}{z}\right)$$

Solution: The claim is true. The idea is to transform the inequality in the antecedent, $w/x < y/z$, into the inequalities in the consequent by multiplying both sides by positive numbers (this preserves the inequality). To keep the argument clear, be sure to work in a single direction: from the antecedent to the consequent.

Exercise 3-2:

PROVE OR DISPROVE: For all quadruples of positive real numbers w, x, y, z, If $w/x < y/z$ then:

$$\left(\frac{w}{x} < \frac{w+y}{x+z}\right) \wedge \left(\frac{w+y}{x+z} < \frac{y}{z}\right)$$

Solution: The claim is true.

Assume w, x, y, z are positive real numbers. # in order to introduce $\forall$

Assume $w/x < y/z$. # in order to introduce $\Rightarrow$

Then $wz < yx$.

multiply both sides of $w/x < y/z$ by $zx \in \mathbb{R}^+$, since $z, x \in \mathbb{R}^+$

Then $wz + yz < yx + yz$. # add yz to both sides

Then $(w+y)/(x+z) < y/z$. # first part of conjunction

multiply both sides by $1/(z(x+z)) \in \mathbb{R}^+$, since $z, (x+z) \in \mathbb{R}^+$.

Exercise 3-3:

PROVE OR DISPROVE: For all quadruples of positive real numbers w, x, y, z, If $w/x < y/z$ then:

$$\left(\frac{w}{x} < \frac{w+y}{x+z}\right) \wedge \left(\frac{w+y}{x+z} < \frac{y}{z}\right)$$

Solution: The claim is true.

Assume w, x, y, z are positive real numbers. # in order to introduce $\forall$

Assume $w/x < y/z$.

...

Then $(w + y)/(x + z) < y/z$. # first part of conjunction

multiply both sides by $1/(z(x + z)) \in \mathbb{R}^+$, since $z, (x + z) \in \mathbb{R}^+$.

Now, add wx to both sides of $wz < yx$ (already established above), yielding $wx + wz < wx + yx$.

Then $w/x < (w + y)/(x + z)$. # multiply both sides by $1/x(x + z)$

Exercise 3-4:

PROVE OR DISPROVE: For all quadruples of positive real numbers w, x, y, z, If $w/x < y/z$ then:

$$\left(\frac{w}{x} < \frac{w+y}{x+z}\right) \wedge \left(\frac{w+y}{x+z} < \frac{y}{z}\right)$$

Solution: The claim is true.

Assume w, x, y, z are positive real numbers. # in order to introduce $\forall$

Assume $w/x < y/z$. Then $wz < yx$.

...

Then $(w+y)/(x+z) < y/z$. # first part of conjunction

...

Then $w/x < (w+y)/(x+z)$. # multiply both sides by $1/x(x+z)$

Then $w/x < (w+y)/(x+z) \wedge (w+y)/(x+z) < y/z$. # introduced conjunction

Then $w/x < y/z \Rightarrow w/x < (w+y)/(x+z) \wedge (w+y)/(x+z) < y/z$. # introduced $\Rightarrow$

Then $\forall w, x, y, z \in \mathbb{R}^+, w/x < y/z \Rightarrow w/x < (w+y)/(x+z) \wedge (w+y)/(x+z) < y/z$.

Exercise 4-1:

PROVE OR DISPROVE: For every pair of positive natural numbers (m, n), if $m \geq n$, then the $\gcd(m, n) = \gcd(n, m - n)$. Your proof/disproof must use the following definition of the $\gcd$ (Greatest Common Divisor)

$\gcd(m, n)$

"greatest common divisor of m and n." The largest positive integer that divides both m and n.

Solution: The claim is true.
The idea is to use the properties of the $\gcd$ from the definition — that it divides both numbers, and is the largest integer that does so. By showing that $\gcd(m, n)$ is a common factor of $(n, m - n)$, and that $\gcd(n, m - n)$ is a common factor of (m, n), you can use the part of the definition about being the GREATEST common factor. . . in two directions.

Exercise 4: PROVE for every pair of positive natural numbers (m, n), if $m \geq n$, then the $\gcd(m, n) = \gcd(n, m - n)$.

Assume m, n are positive natural numbers. # in order to introduce $\forall$

  Assume $m \geq n$. # in order to introduce $\Rightarrow$

  ...

  Then $m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\Rightarrow$

Then $\forall m, n \in \mathbb{N}^+, m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\forall$

Exercise 4: PROVE for every pair of positive natural numbers (m, n), if $m \geq n$, then the $\gcd(m, n) = \gcd(n, m - n)$.

Assume m, n are positive natural numbers. # in order to introduce $\forall$

Assume $m \geq n$. # in order to introduce $\Rightarrow$

Pick $g_1 = \gcd(m, n), g_2 = \gcd(n, m - n)$. # definition and convenience of denotion

Then $\exists i, j \in \mathbb{N}, m = ig_1, n = jg_1, i \geq j$. # definition of $g_1 = \gcd(m, n)$.

Then $m - n = (i - j)g_1 .\# i - j \in \mathbb{N}$, $m - n$ factoring g_1

Then g_1 divides both n and $m - n$. # by definition of divides

Then $g_2 \geq g_1$. # definition of $g_2 = \gcd(n, m - n)$.

...

Then $\gcd(m, n) = \gcd(n, m - n)$. # $g_1 = \gcd(m, n)$, $g_2 = \gcd(n, m - n)$.

Then $m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\Rightarrow$

Then $\forall m, n \in \mathbb{N}, m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\forall$

Exercise 4:PROVE for every pair of positive natural numbers (m, n), if $m \geq n$, then the $\gcd(m, n) = \gcd(n, m - n)$.

Assume m, n are positive natural numbers. # in order to introduce $\forall$

Assume $m \geq n$. # in order to introduce $\Rightarrow$

Pick $g_1 = \gcd(m, n), g_2 = \gcd(n, m - n)$. # definition and convenience of denotion

Then $\exists i, j \in \mathbb{N}, m = ig_1, n = jg_1$. # definition of $g_1 = \gcd(m, n)$.

...

Then $g_2 \geq g_1$. # definition of $g_2 = \gcd(n, m - n)$.

Then $\exists k, l \in \mathbb{N}, k \geq l, n = kg_2, (m - n) = lg_2$. # definition of $g_2 = \gcd(n, m - n)$.

Then $m = (k + l)g_2$ # substitute m, n with factor of g_2, $(k + l) \in \mathbb{N}$.

Then g_2 divides both m and n. # by definition of divides

Then $g_1 \geq g2$. # definition of $g_1 = \gcd(m, n)$.

Then $g_1 \geq g_2 \wedge g_2 \geq g1$. # conjunction

Then $g_1 = g_2$. # algebra

Then $\gcd(m, n) = \gcd(n, m - n)$. # $g_1 = \gcd(m, n)$, $g_2 = \gcd(n, m - n)$.

Then $m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\Rightarrow$

Then $\forall m, n \in \mathbb{N}, m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\forall$