Chapter 3

Formal Proofs

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Announcements

- **Term Test 1**:
  - Section **L0101**: Tuesday **Feb 03, 2:10-3:30** Location: **MP203**
  - Section **L0201**: Thursday **Feb 05, 2:10-3:30** Location: **MP103**
  - You **must** write the quiz in the section that you are enrolled in, unless you have talked to the instructors and they allowed you to switch.
  - **Content: Chapter 2**. Review **lecture** and **course** notes!
- **TA office Hours**:
  - Monday, Feb 02, 1-3pm, 4:30-6:30pm in **BA3201**
  - Wednesday, Feb 04, 12-2pm, 3:30-5:30pm in **BA3201**

Today's Topics

- **Direct Proof of the Existential**
- **Proof of Multiple Quantifiers, Implications, and Conjunctions**
  - **Example of Proving a Statement about a Sequence**
  - **Example of Disproving a Statement about a Sequence**

Chapter 3

Formal Proofs

Direct Proof of the Existential

Direct Proof of the Existential

General Form

- **Prove**: $\exists x \in D, P(x)$.
- **How to prove**:
  Find **one** element in D that **satisfies** P.

Structure for the Direct Proof of Existential

Let $\mathbf{x} = \ldots$ # choose a particular element of the domain
Then $x \in D$. # this **may be obvious**, otherwise prove it
$\vdots$ # prove $P(x)$
Then $\mathbf{P(x)}$. # you've shown that **x satisfies P**
$\exists x \in D, P(x)$. # introduce existential

Direct Proof of the Existential

Exercise

- Prove $\exists x \in \mathbb{R}, x^2 + 2x + 1 = 0$.

  Let $\mathbf{x = -1}$. # choose a particular element that will work
  Then $x \in \mathbb{R}$. # since $-1 \in \mathbb{R}$
  Then $x^2 + 2x + 1 = (-1)^2 + 2(-1)^2 + 1 = 1 - 2 + 1 = 0$. # substitute -1 for x
  Then $\exists x \in \mathbb{R}, x^2 + 2x + 1 = 0$. # we gave an example, so we introduce existential

Direct Proof of the Existential

Exercise

- Prove $\exists x \in \mathbb{Z}, x = -x$.

  Let $\mathbf{x} = \mathbf{0}$. # choose a particular element that will work
  Then $x \in \mathbb{Z}$. # since $0 \in \mathbb{Z}$
  Then $x = 0 = -0 = -x$. # substitute 0 for x
  Then $\exists x \in \mathbb{Z}, x = -x$. # we gave an example, so we introduce existential

Chapter 3

Formal Proofs

Proof of Multiple Quantifiers, Implications, and Conjunctions

Proving a Statement about a Sequence

- $\mathbf{A_1}: a_i = i^2, i \in \mathbb{N}$.

i	0	1	2	3	4	5	...
a_i	0	1	4	9	16	25	...

- $\mathbf{C_1}: \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Verify if $\mathbf{C_1}$ is **True** for $\mathbf{A_1}$.

- $i = 0$: $a_0 = 0^2 \leq 0$, but $0 \not< 0$.
- $i = 1$: $a_1 = 1^2 \leq 1$, but $1 \not< 1$.
- $i = 2$:
  - $a_0 = 0^2 \leq 2$, and $0 < 2$.
  - $a_1 = 1^2 \leq 2$, and $1 < 2$.
  - For all $j \geq 2$, $a_j = j^2 \nleq 2$.

Proving a Statement about a Sequence

- **A_1** : $a_i = i^2, i \in \mathbb{N}$.
- **C_1** : $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that **C_1** is **True** for **A_1**.

Reminder: Direct Proof of Universally Quantified Implication

Structure of an Direct Proof

- Prove $\forall x \in D, P(x) \Rightarrow Q(x)$

Assume $x \in D$. # x is a typical element of D

Assume $P(x)$. # x has property P, the antecedent

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Then $Q(x)$. # the **consequence**!

Then $P(x) \Rightarrow Q(x)$. # assuming antecedent leads to consequent

Then $\forall x \in D, P(x) \Rightarrow Q(x)$. # x was a typical element of D

Reminder: Indirect Proof of Universally Quantified Implication

Structure of an Indirect Proof

- Prove $\forall x \in D, P(x) \Rightarrow Q(x)$

Assume $x \in D$. # x is a typical element of D

Assume $\neg Q(x)$. # negation of the **consequent**!

$\vdots$

Then $\neg P(x)$. # negation of the **antecedent**!

Then $\neg Q(x) \Rightarrow \neg P(x)$. # assuming $\neg Q(x)$ leads to $\neg P(x)$

Then $P(x) \Rightarrow Q(x)$. # implication is equivalent to contrapositive

Then $\forall x \in D, P(x) \implies Q(x)$. # x was a typical element of D

Proving a Statement about a Sequence

- $\mathbf{A_1}$: $a_i = i^2, i \in \mathbb{N}$.
- $\mathbf{C_1}$: $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that $\mathbf{C_1}$ is **True** for $\mathbf{A_1}$.

  Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$

  $\vdots$

  Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proving a Statement about a Sequence

- $\mathbf{A_1}$: $a_i = i^2, i \in \mathbb{N}$.
- $\mathbf{C_1}$: $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that $\mathbf{C_1}$ is **True** for $\mathbf{A_1}$.

  Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$
  Assume $j \in \mathbb{N}$. # typical element of $\mathbb{N}$

  $\vdots$

  Then $\forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce universal
  Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proving a Statement about a Sequence

- $\mathbf{A_1}$: $a_i = i^2, i \in \mathbb{N}$.
- $\mathbf{C_1}$: $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that $\mathbf{C_1}$ is **True** for $\mathbf{A_1}$.

Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$
Assume $j \in \mathbb{N}$. # typical element of $\mathbb{N}$

$\vdots$

Then $a_j \leq 2 \implies j < i$. #
Then $\forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce universal
Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proving a Statement about a Sequence

- $\mathbf{A_1}$: $a_i = i^2, i \in \mathbb{N}$.
- $\mathbf{C_1}$: $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that $\mathbf{C_1}$ is **True** for $\mathbf{A_1}$.

Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$
Assume $j \in \mathbb{N}$. # typical element of $\mathbb{N}$

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Then $\neg(j < i) \implies \neg(a_j \leq 2)$. #
Then $a_j \leq 2 \implies j < i$. # impl. equivalent to contrapos.
Then $\forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce universal
Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proving a Statement about a Sequence

- **A_1** : $a_i = i^2, i \in \mathbb{N}$.
- **C_1** : $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that **C_1** is **True** for **A_1**.

Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$
Assume $j \in \mathbb{N}$. # typical element of $\mathbb{N}$
    Assume $\neg(j < i)$. # antecedent for contrapositive
            $\vdots$
        Then $a_j > 2$. #
    Then $\neg(j < i) \implies \neg(a_j \leq 2)$. # anteced. leads to conseq.
    Then $a_j \leq 2 \implies j < i$. # impl. equivalent to contrapos.
Then $\forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce universal
Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proving a Statement about a Sequence

- $\mathbf{A_1}$: $a_i = i^2, i \in \mathbb{N}$.
- $\mathbf{C_1}$: $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that $\mathbf{C_1}$ is **True** for $\mathbf{A_1}$.

Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$
Assume $j \in \mathbb{N}$. # typical element of $\mathbb{N}$
    Assume $\neg(j < i)$. # antecedent for contrapositive
        Then $j \geq 2$. # negation of $j < i$ when $i = 2$
        $\vdots$
        Then $a_j > 2$. #
    Then $\neg(j < i) \implies \neg(a_j \leq 2)$. # anteced. leads to conseq.
    Then $a_j \leq 2 \implies j < i$. # impl. equivalent to contrapos.
Then $\forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce universal
Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proving a Statement about a Sequence

- **A_1** : $a_i = i^2, i \in \mathbb{N}$.
- **C_1** : $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$.

- Prove that **C_1** is **True** for **A_1**.

Let $i = 2$. Then $i \in \mathbb{N}$. # $2 \in \mathbb{N}$
Assume $j \in \mathbb{N}$. # typical element of $\mathbb{N}$
  Assume $\neg(j < i)$. # antecedent for contrapositive
    Then $j \geq 2$. # negation of $j < i$ when $i = 2$
    Then $a_j = j^2 \geq 2^2 = 4$. # since $a_j = j^2$, and $j \geq 2$
    Then $a_j > 2$. # since $4 > 2$
  Then $\neg(j < i) \implies \neg(a_j \leq 2)$. # anteced. leads to conseq.
  Then $a_j \leq 2 \implies j < i$. # impl. equivalent to contrapos.
Then $\forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce universal
Then $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i$. # introduce existential

Proof of Multiple Quantifiers

Structure

- Prove $\forall x \in D, \exists y \in E, P(x, y)$

Assume $x \in D$. # x is a typical element of D

    Let $y =$ ____. # choose a particular element of the domain

    Then $y \in E$. # this may be obvious, otherwise prove it

    $\vdots$ # prove $P(x, y)$

    Then $P(x, y)$.

    Then $\exists y \in E, P(x, y)$. # introduce existential

Then $\forall x \in D, \exists y \in E, P(x, y)$. # introduce universal

Proof of Multiple Quantifiers

Structure

- Prove $\exists x \in D, \forall y \in E, P(x, y)$

Let $x =$ ____. # choose a particular element of the domain
Then $x \in D$. # this may be obvious, otherwise prove it
Assume $y \in E$. # y is a typical element of E

$\vdots$ # prove $P(x, y)$

Then $P(x, y)$.
Then $\forall x \in D, P(x, y)$. # introduce universal
Then $\exists y \in E, \forall x \in D, P(x, y)$. # introduce existential

Disproving a Statement about a Sequence

- **A₂** : $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- **C_2** : $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$.
- **How to disprove C_2?**
  Prove $\neg C_2$
- $\neg C_2$: $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

Disproving a Statement about a Sequence

- $\mathbf{A_2}$: $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- $\neg\mathbf{C_2}$: $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

- Prove that $\mathbf{C_2}$ is **False** for $\mathbf{A_2}$.

  Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$

  $\vdots$

  Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of universal

Disproving a Statement about a Sequence

- $\mathbf{A_2}$: $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- $\neg\mathbf{C_2}$: $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

- Prove that $\mathbf{C_2}$ is **False** for $\mathbf{A_2}$.

Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$

  Let $j =$ ____. Then $j \in \mathbb{N}$.

  ⋮

  Then $\exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of existential

Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of universal

Disproving a Statement about a Sequence

- $\mathbf{A_2}$: $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- $\neg\mathbf{C_2}$: $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

- Prove that $\mathbf{C_2}$ is **False** for $\mathbf{A_2}$.

Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$

Let $j =$ ____. Then $j \in \mathbb{N}$.

$\vdots$

Then $j > i \wedge a_j \neq a_i$.

Then $\exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of existential

Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of universal

Disproving a Statement about a Sequence

- $\mathbf{A_2}$: $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- $\neg\mathbf{C_2}$: $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i$

- Prove that $\mathbf{C_2}$ is **False** for $\mathbf{A_2}$.

Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$
    Let $j =$ ____. Then $j \in \mathbb{N}$.
        ⋮
    Then $j > i$.
        ⋮
    Then $a_j \neq a_i$.
    Then $j > i \land a_j \neq a_i$. # introduction of conjunction
    Then $\exists j \in \mathbb{N}, j > i \land a_j \neq a_i$. # introduction of existential
Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i$. # introduction of universal

Disproving a Statement about a Sequence

- **A₂** : $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- $\neg$**C₂** : $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

- Prove that **C₂** is **False** for **A₂**.

Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$

  Let $j = i + 2$. Then $j \in \mathbb{N}$. # $i, 2 \in \mathbb{N}$, and $\mathbb{N}$ is closed under $+$

  Then $j > i$. # $2 > 0$, so $i + 2 > i$

  ⋮

  Then $a_j \neq a_i$.

  Then $j > i \wedge a_j \neq a_i$. # introduction of conjunction Then

  $\exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of existential

Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of universal

Disproving a Statement about a Sequence

- **A_2** : $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

i	0	1	2	3	4	5	6	7	8	9	10	...
a_i	0	0	1	1	2	2	3	3	4	4	5	...

- **$\neg C_2$** : $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

- Prove that **C_2** is **False** for **A_2**.

Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$
  Let $j = i + 2$. Then $j \in \mathbb{N}$. # $i, 2 \in \mathbb{N}$, and $\mathbb{N}$ is closed under $+$
  Then $j > i$. # $2 > 0$, so $i + 2 > i$
  Then $a_j = a_{i+2} = a_i + 1$ # since $j \geq 2$ and by Def. of A_2
  Then $a_j \neq a_i$. # $1 > 0$, so $a_i + 1 > a_i$
  Then $j > i \wedge a_j \neq a_i$. # introduction of conjunction Then
  $\exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of existential
Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$. # introduction of universal

Proof of Conjunction

Structure

- Prove $\forall x \in D, P(x) \wedge Q(x)$

  Assume $x \in D$. # x is a typical element of D

  $\vdots$ # prove $P(x)$

  Then $P(x)$.

  $\vdots$ # prove $Q(x)$

  Then $Q(x)$.

  Then $P(x) \wedge Q(x)$. # introduce conjunction

  Then $\forall x \in D, P(x) \wedge Q(x)$. # introduce universal