Chapter 2

Logical Notation

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Announcements

- **Term Test 1**:
  - Time: Tuesday **Feb 03, 2:10-3:30**
  - Location: **MP203**
  - **Chapter 2**
  - **TA office Hours**: TBA
- **Assignment 1** is due on Friday Jan 30, before midnight.
- **TA office Hours** for Assignment 1:
  - Tuesday, Jan 27, 5-7pm in **BA3201**
  - Thursday, Jan 29, 3:30-5:30pm in **BA3201**

Today's Topics

- **Review: Venn Diagram, Sets and Predicates**
- **Review: Logical Symbols, Translation between Logic and English**
- **Review: Logical Grammar and Arithmetic**
- **Review: Truth Tables**
- **Review: Application of Manipulation Rules**

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Logical Notation

Review: Venn Diagram, Sets and Predicates

Visualizing Relationships between Sets

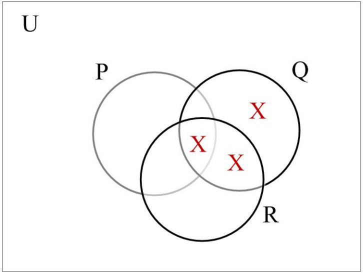
Venn Diagram

- The **rectangle** represents the **domain**.
- Each **circle** represent **a set** in the domain.
- **O** in a part of a set means that this part must be **occupied**, i.e., there must be some element in there.
- **X** in a part of a set means that this part must be **empty**, i.e., contains no element.
- **Important**: Do NOT put anything in regions that are **not specified** as empty or occupied.

Visualizing Relationships between Sets

S3: All quixotic humans are pernicious, but not raffish.

- U denotes the set of all humans,
  P denotes the set of pernicious humans,
  Q denotes the set of quixotic humans, and
  R denotes the set of raffish humans.
- $Q \subseteq P, Q \cap R = \varnothing$



Review: Sets and Predicates

Predicates

An n-ary predicate $L(x_1, ..., x_n)$ is a **boolean function** returning **True** or **False** such that
$L(x_1, ..., x_n) =$ **True** iff $\langle x_1, ..., x_n \rangle$ **satisfies** the property that is denoted by L.

Important Notes about Predicates

- $L(x)$ is **not a set**!
- In a logical statement, you cannot treat a symbol both as a set and a predicate symbol
  - **Incorrect** use of notation: $\forall x, y \in E$, $x \in L, L(y)$.
  - **Correct** version: $\forall x, y \in E$, $x \in L, y \in L$ or $\forall x, y \in E, L(x) \wedge L(y)$.
- Don't apply **set operations over predicates**!
  $\mathbf{P}(\mathbf{x}) \cap \mathbf{Q}(\mathbf{y})$ makes no sense
- Don't apply **logical symbols over sets**!
  $\mathbf{x} \in \mathbf{S} \wedge \mathbf{x} \in \mathbf{R}$ makes no sense
- Don't **nest** predicates!
  $\mathbf{P}(\mathbf{Q}(\mathbf{x}))$ makes no sense

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Review: Logical Symbols, Translation between Logic and English

Evaluating Statements

$\neg P$ is **True** iff P is **False**.

$P \wedge Q$ is **True** iff both P and Q are **True**.

$P \vee Q$ is **True** iff **at least** one of P or Q are **True**.

$P \Rightarrow Q$ is **True** iff **at least** one of $\neg P$ or Q are **True**.

$P \Leftrightarrow Q$ is **True** iff the **truth** values of P and Q are **equal**.

$\forall x \in D, S(x)$ is **True** iff **all** elements of D satisfy the property denoted by S.

$\exists x \in D, S(x)$ is **True** iff **at least one** element of D satisfies the property denoted by S.

Logical Notation and English

Order of Quantifiers matter!

1. $\forall x \in Men, \exists y \in Women, wife(y, x)$
2. $\exists x \in Men, \forall y \in Women, wife(y, x)$

1. All men have wives.
2. There exists a man such that all women are his wives!

Expressing Uniqueness

- All men have **exactly one** wife.
  All men have at least one wife, **and**, All men have at most one wife.
- $\forall x \in Men, \exists y \in Women, wife(y, x) \wedge \forall z \in Women, wife(z, x) \Rightarrow z = y$

Expressing Equality

- There are at least two **different** students who passed the **same** courses.
- $\exists x \in S, \exists y \in S, \forall c \in C, (x \neq y) \wedge (pass(x, c) \Leftrightarrow pass(y, c))$

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Review: Logical Grammar and Arithmetic

Well-Formed Formulas

Well-Formed Formulas (wff)

- Any predicate is a **wff**.
- If P is a **wff**, so is $\neg P$.
- If P and Q are **wffs**, so is $(P \wedge Q)$.
- If P and Q are **wffs**, so is $(P \vee Q)$.
- If P and Q are **wffs**, so is $(P \Rightarrow Q)$.
- If P and Q are **wffs**, so is $(P \Leftrightarrow Q)$.
- If P is a **wff** (possibly open in variable x) and D is a set, then $(\forall x \in D, P)$ is a **wff**.
- If P is a **wff** (possibly open in variable x) and D is a set, then $(\exists x \in D, P)$ is a **wff**.
- Nothing else is a **wff**.

Precedence decreases from top to bottom.

Logical Arithmetic

Commutative

- $P \wedge Q \Leftrightarrow Q \wedge P$
- $P \vee Q \Leftrightarrow Q \vee P$

Associative

- $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$
- $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

Distributive

- $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Identity

- $P \wedge (Q \vee \neg Q) \Leftrightarrow P \Leftrightarrow P \vee (Q \wedge \neg Q)$

Idempotency

- $P \wedge P \Leftrightarrow P \Leftrightarrow P \vee P$

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Review: Truth Tables

Truth Tables

Truth Tables

In a **truth table**, we write **all** possible truth values for the **predicates** in a statement and compute the truth value of the statement under **each** of these truth assignments.

Truth Tables for Logical Symbols

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Truth Tables

Evaluate S_1

$\mathbf{S_1} : (\neg P \Rightarrow (\neg Q \Rightarrow \neg R)) \Leftrightarrow (R \Rightarrow (P \vee Q))$
$\mathbf{S_2} : \neg P \Rightarrow (\neg Q \Rightarrow \neg R)$
$\mathbf{S_3} : R \Rightarrow (P \vee Q)$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$\neg Q \Rightarrow \neg R$	S_2	$P \vee Q$	S_3	$\mathbf{S_1}$
T	T	T	F	F	F	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T	T
T	F	T	F	T	F	F	T	T	T	T
T	F	F	F	T	T	T	T	T	T	T
F	T	T	T	F	F	T	T	T	T	T
F	T	F	T	F	T	T	T	T	T	T
F	F	T	T	T	F	F	F	F	F	T
F	F	F	T	T	T	T	T	F	T	T

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Review: Application of Manipulation Rules

Application of Manipulation Rules

1. Use manipulation Rules to show that $(P \vee Q) \wedge (P \vee \neg Q)$ is equal to P

$$
\begin{aligned}
(P \vee Q) \wedge (P \vee \neg Q) &\Leftrightarrow P \vee (Q \wedge \neg Q) \quad \text{\# distributive } \vee \text{ over } \wedge \\
&\Leftrightarrow P \quad \text{\# Identity law}
\end{aligned}
$$

2. Use manipulation Rules to show that the following statement is a tautology $((P \wedge Q) \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$

$$
\begin{aligned}
((P \wedge Q) \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R)) &\Leftrightarrow \neg(\neg(P \wedge Q) \vee R) \vee (\neg P \vee (\neg Q \vee R))\text{\# implication} \\
&\Leftrightarrow \neg((\neg P \vee \neg Q) \vee R) \vee (\neg P \vee (\neg Q \vee R))\text{\# DeMorgan's} \\
&\Leftrightarrow \neg((\neg P \vee \neg Q) \vee R) \vee ((\neg P \vee \neg Q) \vee R)\text{\# Associative} \\
&\Leftrightarrow \textbf{True} \quad \text{\# Definition of } \vee
\end{aligned}
$$