CSC165 Mathematical Expression and Reasoning for Computer Science

Lisa Yan

Department of Computer Science
University of Toronto

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Announcements

- Term Test 1:
  - Time: Thursday Feb 05, 2:10-3:30
  - Location: MP103
  - Chapter 2
  - TA office Hours: TBA
- Assignment 1 is due on Friday Jan 30, before midnight.
- TA office Hours for Assignment 1:
  - Tuesday, Jan 27, 5-7pm in BA3201
  - Thursday, Jan 29, 3:30-5:30pm in BA3201

Chapter 2 Reviewing Session

- Sets and Predicates
- Equivalence
- Tautology/Satisfiable/Unsatisfiable
- Logical Grammar
- Logical Arithmetic
- Manipulation Rules
- Truth Tables

Sets and Predicates

Predicates

An n-ary predicate $L(x_1, ..., x_n)$ is a BOOLEAN FUNCTION returning $\mathbf{True}$ or $\mathbf{False}$ such that
$L(x_1, ..., x_n) = \mathbf{True}$ iff $\langle x_1, ..., x_n \rangle$ SATISFIES the property that is denoted by L.

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Important Notes about Predicates

- $L(x)$ is NOT A SET!
- In a logical statement, you cannot treat a symbol both as a set and a predicate symbol
  - INCORRECT use of notation: $\forall x, y \in E, x \in L, L(y)$.
  - CORRECT version: $\forall x, y \in E, x \in L, y \in L$ or $\forall x, y \in E, L(x) \land L(y)$.

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- Don't apply SET OPERATIONS OVER PREDICATES!
  $\mathbf{P}(\mathbf{x}) \cap \mathbf{Q}(\mathbf{y})$ makes no sense
- Don't apply LOGICAL SYMBOLS OVER SETS!
  $\mathbf{x} \in \mathbf{S} \wedge \mathbf{x} \in \mathbf{R}$ makes no sense

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- Don't NEST predicates!
  $\mathbf{P}(\mathbf{Q}(\mathbf{x}))$ makes no sense

Equivalence

Definition: Equivalence

- $P \rightarrow Q$, and conversely $P \leftarrow Q$.
- P iff Q ("iff" being an abbreviation for "if and only if").
- P is necessary and sufficient for Q.
  - $P \rightarrow Q$: P is sufficient for Q.
  - $\neg P \rightarrow \neg Q$: P is necessary for Q.

Equivalence Exercises:

$P \Rightarrow (Q \Rightarrow R) \iff (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$

SAMPLE SOLUTION:

$$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) \iff \neg(P \Rightarrow Q) \vee (\neg P \vee R)$$

(implication)

$$\iff (P \wedge \neg Q) \vee (\neg P \vee R)$$

(implication negation)

$$\iff \big((P \vee \neg P) \wedge (\neg Q \vee \neg P)\big) \vee R$$

(distributivity)

$$\iff (\neg P \vee \neg Q) \vee R$$

(identity and commutativity)

$$\iff \neg P \vee (\neg Q \vee R)$$

(associativity)

$$\iff P \Rightarrow (Q \Rightarrow R)$$

(implication)

TAUTOLOGY/SATISFIABLE/UNSATISFIABLE

Definition: TAUTOLOGY

No domain, or a meaning for predicates P and Q that provides a counter-example, since the truth tables are identical.

Definition: SATISFIABLE

True for some choice of domain, predicates P and Q, and values of domain elements, so we say this statement is SATISFIABLE.
But in this case, there are also choices of domains and/or predicates in which it is false, so it is not a tautology.

Definition: UNSATISFIABLE (or CONTRADICTION)

No domains, predicates, or values can be chosen to make it true, such a statement would be UNSATISFIABLE (or a CONTRADICTION).

Exercises

- Use TRUTH TABLES to evaluate the following claims. Indicate which one is a TAUTOLOGY, which one is SATISFIABLE and which one is UNSATISFIABLE.
  1. $P \Rightarrow Q$ is equivalent to its contrapositive.
     T
  2. $P \Rightarrow Q$ is equivalent to its converse.
     S
  3. $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
     T
  4. $P \wedge \neg P$.
     U
  5. $P \vee \neg P$.
     T

Logical Grammar

Well-Formed Formula (wff): a syntactically correct sentence

The syntax (or grammar rules) can be summarized as follows:

- Any predicate P, Q is a wff.
- $\neg P$.
- $(P \wedge Q)$.
- $(P \vee Q)$.
- $(P \Rightarrow Q)$.
- $(P \Leftrightarrow Q)$.
- $(\forall x \in D, P)$ is a wff.
- $(\exists x \in D, P)$ is a wff.
- Nothing else is a wff.

Logical Grammar Exercises:

Are the following sentences WFF?

1. $R(\forall x \in E, y)$.
2. $\exists x \in E, \exists y \in E$.
3. $\exists x \in E \wedge \exists y \in E$.
4. $R(x, y)$.
5. $\exists x \in E, R(x), S(x)$.
6. $\forall x \in U, \exists y \in E, \forall z \in U, R(x, y) \vee S(x, z)$.

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Logical Arithmetic

Precedence Rules

Precedence DECREASES from TOP TO BOTTOM:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

Arithmetic Rules

1. Commutative
2. Associative
3. Distributive
4. Identity
5. Idempotency
6. DeMorgan's Law
7. etc.

Logical Arithmetic

Commutative

$P \wedge Q \Leftrightarrow Q \wedge P$ and $P \vee Q \Leftrightarrow Q \vee P$

Associative

$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ and $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

Distributive

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ and $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Identity

$P \wedge (Q \vee \neg Q) \Leftrightarrow P \Leftrightarrow P \vee (Q \wedge \neg Q)$

Idempotency

$P \wedge P \Leftrightarrow P \Leftrightarrow P \vee P$

Arithmetic Exercises:

Standard Equivalences: P, Q, $P(x)$, $Q(x)$ are arbitrary sentences

- *Commutativity*
  $P \land Q \iff Q \land P$
  $P \lor Q \iff Q \lor P$
  $P \Leftrightarrow Q \iff Q \Leftrightarrow P$
- *Associativity*
  $P \land (Q \land R) \iff (P \land Q) \land R$
  $P \lor (Q \lor R) \iff (P \lor Q) \lor R$
- *Identity*
  $P \land (Q \lor \neg Q) \iff P$
  $P \lor (Q \land \neg Q) \iff P$
- *Absorption*
  $P \land (Q \land \neg Q) \iff Q \land \neg Q$
  $P \lor (Q \lor \neg Q) \iff Q \lor \neg Q$
- *Idempotency*
  $P \land P \iff P$
  $P \lor P \iff P$
- *Double Negation*
  $\neg\neg P \iff P$
- *DeMorgan's Laws*
  $\neg(P \land Q) \iff \neg P \lor \neg Q$
  $\neg(P \lor Q) \iff \neg P \land \neg Q$

Arithmetic Exercises:

Standard Equivalences: P, Q, $P(x)$, $Q(x)$ are arbitrary sentences

- *Distributivity*
  $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- *Implication*
  $P \Rightarrow Q \iff \neg P \vee Q$
- *Biconditional*
  $P \Leftrightarrow Q \iff (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- *Renaming* (where $P(x)$ does not contain variable y)
  $\forall x, P(x) \iff \forall y, P(y)$
  $\exists x, P(x) \iff \exists y, P(y)$
- *Quantifier Negation*
  $\neg \forall x, P(x) \iff \exists x, \neg P(x)$
  $\neg \exists x, P(x) \iff \forall x, \neg P(x)$
- *Quantifier Commutativity*
  $\forall x, \forall y, S(x,y) \iff \forall y, \forall x, S(x,y)$
  $\exists x, \exists y, S(x,y) \iff \exists y, \exists x, S(x,y)$
- *Quantifier Distributivity* (where S does not contain variable x)
  $S \wedge \forall x, Q(x) \iff \forall x, S \wedge Q(x)$
  $S \vee \forall x, Q(x) \iff \forall x, S \vee Q(x)$
  $S \wedge \exists x, Q(x) \iff \exists x, S \wedge Q(x)$
  $S \vee \exists x, Q(x) \iff \exists x, S \vee Q(x)$

Manipulation rules - 1

The following is a summary of the basic laws and rules we use for manipulating formal statements.

identity laws	$P \wedge (Q \vee \neg Q)$	$\iff$	P
	$P \vee (Q \wedge \neg Q)$	$\iff$	P
idempotency laws	$P \wedge P$	$\iff$	P
	$P \vee P$	$\iff$	P
commutative laws	$P \wedge Q$	$\iff$	$Q \wedge P$
	$P \vee Q$	$\iff$	$Q \vee P$
	$(P \Leftrightarrow Q)$	$\iff$	$(Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R$	$\iff$	$P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R$	$\iff$	$P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R)$	$\iff$	$(P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R)$	$\iff$	$(P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q$	$\iff$	$\neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q$	$\iff$	$\neg P \vee Q$
equivalence	$(P \Leftrightarrow Q)$	$\iff$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Manipulation rules - 2

double negation	$\neg(\neg P) \iff P$
DeMorgan's laws	$\neg(P \wedge Q) \iff \neg P \vee \neg Q$
	$\neg(P \vee Q) \iff \neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q) \iff P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q) \iff \neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x)) \iff \exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x)) \iff \forall x \in D, \neg P(x)$
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x) \iff (\forall x \in D, P(x)) \wedge (\forall x \in D,$
(where R does not contain variable x)	$\exists x \in D, P(x) \vee Q(x) \iff (\exists x \in D, P(x)) \vee (\exists x \in D,$
	$\forall x \in D, R \wedge Q(x) \iff R \wedge (\forall x \in D, Q(x))$
	$\forall x \in D, R \vee Q(x) \iff R \vee (\forall x \in D, Q(x))$
	$\exists x \in D, R \vee Q(x) \iff R \vee (\exists x \in D, Q(x))$
	$\exists x \in D, R \wedge Q(x) \iff R \wedge (\exists x \in D, Q(x))$
variable renaming	$\forall x \in D, P(x) \iff \forall y \in D, P(y)$
(where y does not appear in $P(x)$)	$\exists x \in D, P(x) \iff \exists y \in D, P(y)$

Manipulation Rules Exercises:

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$$(P \vee Q) \wedge (P \vee \neg Q) \quad \Leftrightarrow$$

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\begin{aligned}
(P \vee Q) \wedge (P \vee \neg Q) &\Leftrightarrow P \vee (Q \wedge \neg Q) && \text{distributive } \vee \text{ over } \wedge \\
&\Leftrightarrow P && \text{Identity law}
\end{aligned}
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$((P \wedge Q) \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R)) \quad \Leftrightarrow \quad \neg(\neg(P \wedge Q) \vee R) \vee (\neg P \vee (\neg Q \vee R))$# implication

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$$\Leftrightarrow \quad \neg((\neg P \vee \neg Q) \vee R) \vee (\neg P \vee (\neg Q \vee R))\text{# DeMorgan's}$$

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$$\Leftrightarrow \quad \mathbf{True} \qquad \text{\# Definition of } \vee$$

Truth Tables

Truth Tables

In a TRUTH TABLE, we write ALL possible truth values for the PREDICATES in a statement and compute the truth value of the statement under EACH of these truth assignments.

Truth Tables for Logical Symbols

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Truth Tables Exercises:

Evaluate S_1

$\mathbf{S_1}: (\neg P \Rightarrow (\neg Q \Rightarrow \neg R)) \Leftrightarrow (R \Rightarrow (P \vee Q))$

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$\mathbf{S_2}$: $\neg P \Rightarrow (\neg Q \Rightarrow \neg R)$
$\mathbf{S_3}$: $R \Rightarrow (P \vee Q)$

Truth Tables Exercises:

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$\mathbf{S_1} : (\neg P \Rightarrow (\neg Q \Rightarrow \neg R)) \Leftrightarrow (R \Rightarrow (P \vee Q))$
$\mathbf{S_2} : \neg P \Rightarrow (\neg Q \Rightarrow \neg R)$
$\mathbf{S_3} : R \Rightarrow (P \vee Q)$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$\neg Q \Rightarrow \neg R$	S_2	$P \vee Q$	S_3	$\mathbf{S_1}$
T	T	T	F	F	F	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T	T
T	F	T	F	T	F	F	T	T	T	T
T	F	F	F	T	T	T	T	T	T	T
F	T	T	T	F	F	T	T	T	T	T
F	T	F	T	F	T	T	T	T	T	T
F	F	T	T	T	F	F	F	F	F	T
F	F	F	T	T	T	T	T	F	T	T