Chapter 3

Formal Proofs

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Announcements

- **Assignment 1** is due Today, before midnight.
- **Term Test 1**:
  - Section **L0101**: Tuesday **Feb 03, 2:10-3:30** Location: **MP203**
  - Section **L0201**: Thursday **Feb 05, 2:10-3:30** Location: **MP103**
  - You **must** write the quiz in the section that you are enrolled in, unless you have talked to the instructors and they allowed you to switch.
  - **Content: Chapter 2**. Review **lecture** and **course** notes!
- **TA office Hours**:
  - Monday, Feb 02, 1-3pm, 4:30-6:30pm in **BA3201**
  - Wednesday, Feb 04, 12-2pm, 3:30-5:30pm in **BA3201**

Today's Topics

- **Review: Proof by Contradiction**

Chapter 3

Formal Proofs

Review: Proof by Contradiction

Proof by Contradiction

Solving a Murder Case

- **Mr. Holmes** has asked his trusty companion **Dr. Watson** to assist him in solving a murder case.
  - *Holmes's* main **suspect** is **Jack the Ripper**.

  Dr. Watson has gathered some **facts** about the murder, and wants to impress Holmes by solving the case.
  However, he can not derive any conclusion by himself.
  So he has asked YOU for help.

  Here's what he has so far:
  - The victim was shot by **a gun**.
  - Jack the Ripper **never uses a gun** on his victims.
- What's you expert opinion for Dr. Watson?
  Jack the Ripper **is not the murderer!**

  You should give a **concrete argument** that convinces *Mr. Holmes*!

Proof by Contradiction

Given P Prove Q

1. Assume $\neg Q$.
2. Use $\neg Q$ to show $\neg P$.
   It means, $\neg Q \Rightarrow \neg P$ is **True**.
3. P is a given **fact**.
4. P and $\neg P$ cannot both be **True**! So $\neg P$ must be **False**.
5. $\neg P$ is **False** and $\neg Q \Rightarrow \neg P$ is **True**.
   So $\neg Q$ is **False** and therefore Q is **True**.

General Structure

Assume $\neg Q$. # in order to derive a contradiction

$\vdots$ # some steps leading to a contradiction, say $\neg P$

Then $\neg P$. # contradiction, since P is **known to be true**

Then Q. # since assuming $\neg Q$ leads to a **contradiction**

Proof by Contradiction

Solving a Murder Case

- Use the **structure for proof by contradiction** to convince Mr. Holmes.
  - The victim was shot by **a gun**. $\rightarrow$ **Fact**
  - Jack the Ripper **never uses a gun** on his victims. $\rightarrow$ **Fact**
- What's you expert opinion for Dr. Watson?
  Jack the Ripper **is not the murderer!** $\rightarrow$ **Conclusion**

Proof by Contradiction

Solving a Murder Case

- P_1: The victim was shot by a gun.
- P_2: Jack the Ripper never uses a gun on his victims.
- Q: Jack the Ripper is not the murderer.

Proof:

Assume Jack the Ripper is the murderer. # to derive a contradiction

Then the Ripper must have used a gun. # because of P_1

Then the Ripper used a gun on one of his victims. # By the previous step ($\neg P_2$)

Then Jack the Ripper is not the murderer. # assuming $\neg Q$ leads to a **contradiction** as P_2 is **known to be true**

Proof by Contradiction

Exercise

- P_1: There are 10 pigeonholes.
- P_2: There are 11 pigeons in the holes in **total**.
- Q: Prove that **at least two** pigeons share a hole.

Proof:

Assume there is at most one pigeon in all holes. # negation of Q

Then there are at most 10 pigeons. # $10 \times 1 = 10$ ($\neg P_2$)

Then **at least two** pigeons share a hole. # since assuming $\neg Q$ leads to a **contradiction**

Proof by Contradiction

Exercise: The Pigeonhole Principle

- If n items are put into m containers, with $n > m$, then at least one container must contain **more than one item**.

Proof:

Assume there is at most one item in all containers. # negation of the **conclusion**

Then there are at most m items. # $m \times 1 = m$ (negation of the assumptions)

Then **at least one** container must contain more than one item. # since assuming the negation of the conclusion leads to a **contradiction**

Proof by Contradiction

Exercise

- Prove that there are infinitely many **even** natural numbers.

Proof:

Assume there are a finite number of even numbers. # negation of the **conclusion**

Then there exists a **largest** even number $n \in \mathbb{N}$. # By the previous step

Let $n' = 2n$. Then n' is an even number, and $n' > n$ # (contradiction, since n is the largest even number)

Then there are **infinitely many** even numbers. # since assuming the negation of the conclusion leads to a **contradiction**