CSC165 Mathematical Expression and Reasoning for Computer Science

Lisa Yan

Department of Computer Science
University of Toronto

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Announcements

- NEXT MONDAY REVIEWING SESSION:
  - Piazza forum: vote for the troublesome topic(s) in Logical Notation;
  - DUE DATE: This Friday before midnight on Piazza.
    - MORNING Section: https://piazza.com/class/i4f701lgr0m75p?cid=56
    - AFTERNOON Section: https://piazza.com/class/i4f701lgr0m75p?cid=57
  - TOPICS:
    - Logical Grammar, Venn Diagram, Predicates, Equivalence
    - Conjunction, Disjunction, Negation, Quantifiers
    - Truth Tables
    - Implication, Logical Arithmetic
    - Manipulation Rules

Today's Topics

- APPLICATION OF NEGATION TO LOGICAL SENTENCES
- LOGICAL ARITHMETIC
- MANIPULATION RULES

Implication rule

$P \Rightarrow Q$ means: Every P is a Q

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P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Evaluate S_1: implication rule

$\mathbf{S_1}$: $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

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Evaluate S_1: implication rule

$\mathbf{S_1} : (P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$	$\mathbf{S_1}$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Exercise #1

- Use TRUTH TABLES to evaluate the following claims. Indicate which one is a TAUTOLOGY, which one is SATISFIABLE and which one is UNSATISFIABLE.
  1. $P \Rightarrow Q$ is equivalent to its contrapositive.
     T
  2. $P \Rightarrow Q$ is equivalent to its converse.
     S
  3. $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \land (Q \Rightarrow P)$.
     T
  4. $P \land \neg P$.
     U
  5. $P \lor \neg P$.
     T

Chapter 2: Logical Notation

Application of Negation to Logical Sentences

Negation

Negation of a Sentence

- The negation of a sentence inverts its TRUTH VALUE.
- The negation of a sentence P is written as $\neg P$,
- $\neg P$ is TRUE if P was FALSE, $\neg P$ is FALSE if P was TRUE.
- $\neg\neg P$ is equal with P.

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- $\neg\neg P$ is equal with P.

Example:

Claim: All employees making over 80,000 are female.

The negation is: Not all employees making over 80,000 are female.

Negation over Conjunction and Disjunction: DeMorgan's Law

- $\neg(S_1 \wedge S_2) \Leftrightarrow (\neg S_1 \vee \neg S_2)$
  Sentence $S_1 \wedge S_2$ is FALSE exactly when at least one of S_1 or S_2 is FALSE.
- $\neg(S_1 \vee S_2) \Leftrightarrow (\neg S_1 \wedge \neg S_2)$
  Sentence $S_1 \vee S_2$ is FALSE exactly when both S_1 and S_2 are FALSE.

Exercise:

Recall that

- $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.
- $(P \Leftrightarrow Q) \Leftrightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$

Now use DeMorgan's law to simplify the following sentences so that only P and Q are negated.

1. $\neg(P \Rightarrow Q)$.
2. $\neg(P \Leftrightarrow Q)$

Negation over Conjunction and Disjunction

Solution:

1. $$\begin{aligned} \neg(P \Rightarrow Q) &\Leftrightarrow \neg(\neg P \vee Q) \\ &\Leftrightarrow (\neg\neg P \wedge \neg Q) \\ &\Leftrightarrow (P \wedge \neg Q) \end{aligned}$$

2. $$\begin{aligned} \neg(P \Leftrightarrow Q) &\Leftrightarrow \neg((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\Leftrightarrow (\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)) \\ &\Leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)) \end{aligned}$$

Negation

Negation over Quantifiers

Example 1:

$\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow \exists x \in D, \neg(\exists y \in D, P(x,y))$
$\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow \exists x \in D, \forall y \in D, \neg P(x,y)$

Example 2:

$\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \neg(\forall y \in D, P(x,y))$
$\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \exists y \in D, \neg P(x,y)$

Example 3:

$\neg(\exists x \in D, \forall y \in D, (P(x,y) \Rightarrow Q(x,y))) \Leftrightarrow \neg(\exists x \in D, \forall y \in D, (\neg P(x,y) \vee Q(x,y)))$

CHAPTER 2: LOGICAL NOTATION

LOGICAL ARITHMETIC

Logical Arithmetic

Precedence Rules

Precedence DECREASES from TOP TO BOTTOM:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

Logical Arithmetic

Precedence Rules

Precedence DECREASES from TOP TO BOTTOM:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

Arithmetic Rules

1. Commutative
2. Associative
3. Distributive
4. Identity
5. Idempotency
6. DeMorgan's Law
7. etc.

These rules can be verified by a truth table.

Logical Arithmetic

Commutative

$P \wedge Q \Leftrightarrow Q \wedge P$ and $P \vee Q \Leftrightarrow Q \vee P$

Logical Arithmetic

Commutative

$P \wedge Q \Leftrightarrow Q \wedge P$ and $P \vee Q \Leftrightarrow Q \vee P$

Associative

$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ and $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

Distributive

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ and $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Logical Arithmetic

Commutative

$P \land Q \Leftrightarrow Q \land P$ and $P \lor Q \Leftrightarrow Q \lor P$

Associative

$P \land (Q \land R) \Leftrightarrow (P \land Q) \land R$ and $P \lor (Q \lor R) \Leftrightarrow (P \lor Q) \lor R$

Distributive

$P \land (Q \lor R) \Leftrightarrow (P \land Q) \lor (P \land R)$ and $P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R)$

Identity

$P \land (Q \lor \neg Q) \Leftrightarrow P \Leftrightarrow P \lor (Q \land \neg Q)$

Idempotency

$P \land P \Leftrightarrow P \Leftrightarrow P \lor P$

Logical Arithmetic: implication, bi-implication, with $\neg$, $\vee$, and $\wedge$

Implication: $P \Rightarrow Q \Leftrightarrow \neg P \vee Q$

Negate implication (use DeMorgan's law):

$$\neg(P \Rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow (\neg\neg P \wedge \neg Q) \Leftrightarrow (P \wedge \neg Q)$$

Bi-implication: $P \Leftrightarrow Q$

Written with $\wedge$, $\vee$, and $\neg$:

$$(P \Leftrightarrow Q) \Leftrightarrow ((\neg P \vee Q) \wedge (\neg Q \vee P)) \Leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

Negate bi-implication (use DeMorgan's law):

$$\neg(P \Leftrightarrow Q) \Leftrightarrow \neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \Leftrightarrow \cdots \Leftrightarrow ((\neg P \wedge Q) \vee (P \wedge \neg Q))$$

Transitivity of Universally-quantified Implication

$((P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))) \Rightarrow (P(x) \Rightarrow R(x))$

$\forall x \in D, (P(x) \Rightarrow (Q(x) \Rightarrow R(x))) \Leftrightarrow \forall x \in D, ((P(x) \wedge Q(x)) \Rightarrow R(x))$

Mixed Multiple Quantifiers

"Everybody has somebody who respects him/her."

...

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"Everybody has somebody who respects him/her."

...

"y is respected by somebody"

$\exists x \in P,\ r(x, y)$

Mixed Multiple Quantifiers

“y is respected by somebody”

$\exists x \in P,\ r(x, y)$

“Everybody has somebody who respects him/her.”

$\forall y \in P, \exists x \in P,\ r(x, y)$

Summary of manipulation rules - 1

The following is a summary of the basic laws and rules we use for manipulating formal statements.

identity laws	$P \wedge (Q \vee \neg Q)$	$\iff$	P
	$P \vee (Q \wedge \neg Q)$	$\iff$	P
idempotency laws	$P \wedge P$	$\iff$	P
	$P \vee P$	$\iff$	P
commutative laws	$P \wedge Q$	$\iff$	$Q \wedge P$
	$P \vee Q$	$\iff$	$Q \vee P$
	$(P \Leftrightarrow Q)$	$\iff$	$(Q \Leftrightarrow P)$
associative laws	$(P \wedge Q) \wedge R$	$\iff$	$P \wedge (Q \wedge R)$
	$(P \vee Q) \vee R$	$\iff$	$P \vee (Q \vee R)$
distributive laws	$P \wedge (Q \vee R)$	$\iff$	$(P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R)$	$\iff$	$(P \vee Q) \wedge (P \vee R)$
contrapositive	$P \Rightarrow Q$	$\iff$	$\neg Q \Rightarrow \neg P$
implication	$P \Rightarrow Q$	$\iff$	$\neg P \vee Q$
equivalence	$(P \Leftrightarrow Q)$	$\iff$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Summary of manipulation rules - 2

double negation	$\neg(\neg P)$	$\Longleftrightarrow$	P
DeMorgan's laws	$\neg(P \wedge Q)$	$\Longleftrightarrow$	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	$\Longleftrightarrow$	$\neg P \wedge \neg Q$
implication negation	$\neg(P \Rightarrow Q)$	$\Longleftrightarrow$	$P \wedge \neg Q$
equivalence negation	$\neg(P \Leftrightarrow Q)$	$\Longleftrightarrow$	$\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$
quantifier negation	$\neg(\forall x \in D, P(x))$	$\Longleftrightarrow$	$\exists x \in D, \neg P(x)$
	$\neg(\exists x \in D, P(x))$	$\Longleftrightarrow$	$\forall x \in D, \neg P(x)$
quantifier distributive laws	$\forall x \in D, P(x) \wedge Q(x)$	$\Longleftrightarrow$	$(\forall x \in D, P(x)) \wedge (\forall x \in D,$
(where R does not contain variable x)	$\exists x \in D, P(x) \vee Q(x)$	$\Longleftrightarrow$	$(\exists x \in D, P(x)) \vee (\exists x \in D,$
	$\forall x \in D, R \wedge Q(x)$	$\Longleftrightarrow$	$R \wedge (\forall x \in D, Q(x))$
	$\forall x \in D, R \vee Q(x)$	$\Longleftrightarrow$	$R \vee (\forall x \in D, Q(x))$
	$\exists x \in D, R \vee Q(x)$	$\Longleftrightarrow$	$R \vee (\exists x \in D, Q(x))$
	$\exists x \in D, R \wedge Q(x)$	$\Longleftrightarrow$	$R \wedge (\exists x \in D, Q(x))$
variable renaming	$\forall x \in D, P(x)$	$\Longleftrightarrow$	$\forall y \in D, P(y)$
(where y does not appear in $P(x)$)	$\exists x \in D, P(x)$	$\Longleftrightarrow$	$\exists y \in D, P(y)$