CSC165 Mathematical Expression and Reasoning for Computer Science

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Announcements

- ASSIGNMENT 1:
  - ASSIGNMENT 1 is posted on the course web page.
  - DUE DATE: Jan 30, before midnight on MarkUs.
  - You WON'T be able to log into MarkUs and submit the assignment BEFORE JAN 24.
  - Assignments may be submitted in groups of up to TWO students. You may choose your group-mate from students in the other section.
  - Submissions must be TYPED. LaTeX is strongly recommended.
  - There are some useful links for LaTeX on the web page.

Today's Topics

- Truth Tables, Tautology, Satisfiability, Unsatisfiability
- Application of Negation to Logical Sentences
- Logical Arithmetic

CHAPTER 2: LOGICAL NOTATION

TRUTH TABLES, TAUTOLOGY SATISFIABILITY, UNSATISFIABILITY

Truth Tables

- Logical statements evaluate either to TRUE or FALSE.
- It's not easy to EVALUATE COMPLEX statements:

$$(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$$

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In a TRUTH TABLE, we write ALL possible truth values for the PREDICATES in a statement and compute the truth value of the statement under EACH of these truth assignments.

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- QUESTION: if there are $\mathbf{n}$ predicates in a statement, how many rows do you need in a truth table to evaluate the statement?

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- QUESTION: if there are $\mathbf{n}$ predicates in a statement, how many rows do you need in a truth table to evaluate the statement? $\mathbf{2^n}$

Truth Tables

Truth Tables for Logical Symbols

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Truth Tables

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P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2} : (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

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Truth Tables for Logical Symbols

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2} : (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\mathbf{S_2}$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

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T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2} : (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\mathbf{S_2}$
T	T	T	T				
T	T	F	F				
T	F	T	T				
T	F	F	T				
F	T	T	T				
F	T	F	F				
F	F	T	T				
F	F	F	T				

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T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2} : (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\mathbf{S_2}$
T	T	T	T	T			
T	T	F	F	F			
T	F	T	T	T			
T	F	F	T	T			
F	T	T	T	T			
F	T	F	F	T			
F	F	T	T	T			
F	F	F	T	T			

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T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2} : (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\mathbf{S_2}$
T	T	T	T	T	T		
T	T	F	F	F	T		
T	F	T	T	T	F		
T	F	F	T	T	F		
F	T	T	T	T	F		
F	T	F	F	T	F		
F	F	T	T	T	F		
F	F	F	T	T	F		

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T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2}$: $(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\mathbf{S_2}$
T	T	T	T	T	T	T	
T	T	F	F	F	T	F	
T	F	T	T	T	F	T	
T	F	F	T	T	F	T	
F	T	T	T	T	F	T	
F	T	F	F	T	F	T	
F	F	T	T	T	F	T	
F	F	F	T	T	F	T	

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T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_2

$\mathbf{S_2} : (P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\mathbf{S_2}$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Truth Tables

Truth Tables for Logical Symbols

P	Q	$\neg P$	$\neg Q$	$P \land Q$	$P \lor Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

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P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_1: implication rule

$\mathbf{S_1} : (P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

Truth Tables

Truth Tables for Logical Symbols

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Evaluate S_1: implication rule

$\mathbf{S_1} : (P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$	$\mathbf{S_1}$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

New Terms

Tautology

A TAUTOLOGY is a sentence that is ALWAYS TRUE in any domain.

Satisfiability

A statement is SATISFIABLE if it is TRUE IN SOME domain / if it is possible to find an interpretation (model) that makes the statement true.

Unsatisfiability (Contradiction)

A statement is UNSATISFIABLE if it is ALWAYS FALSE in any domain domains / if none of the interpretations make the statement true.

Exercise #1

- Use TRUTH TABLES to evaluate the following claims. Indicate which one is a TAUTOLOGY, which one is SATISFIABLE and which one is UNSATISFIABLE.
  1. $P \Rightarrow Q$ is equivalent to its contrapositive.
  2. $P \Rightarrow Q$ is equivalent to its converse.
  3. $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
  4. $P \wedge \neg P$.
  5. $P \vee \neg P$.

Chapter 2: Logical Notation

Application of Negation to Logical Sentences

Negation

Negation of a Sentence

- The negation of a sentence inverts its TRUTH VALUE.
- The negation of a sentence P is written as $\neg P$,
- $\neg P$ is TRUE if P was FALSE, $\neg P$ is FALSE if P was TRUE.
- $\neg\neg P$ is equal with P. (why?)

Negation

Negation of a Sentence

- The negation of a sentence inverts its TRUTH VALUE.
- The negation of a sentence P is written as $\neg P$,
- $\neg P$ is TRUE if P was FALSE, $\neg P$ is FALSE if P was TRUE.
- $\neg\neg P$ is equal with P. (why?)

Example:

Claim: All employees making over 80,000 are female.

The negation is: Not all employees making over 80,000 are female.

Negation over Conjunction and Disjunction

DeMorgan's Law

- Sentence $S_1 \wedge S_2$ is FALSE exactly when at least one of S_1 or S_2 is FALSE.

$$\neg(S_1 \wedge S_2) \Leftrightarrow (\neg S_1 \vee \neg S_2)$$

- Sentence $S_1 \vee S_2$ is FALSE exactly when both S_1 and S_2 are FALSE.

$$\neg(S_1 \vee S_2) \Leftrightarrow (\neg S_1 \wedge \neg S_2)$$

These laws can be VERIFIED either by a truth table, or by representing the sentences as Venn diagrams and taking the complement.

Negation over Conjunction and Disjunction

Exercise:

Recall that

- $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.
- $(P \Leftrightarrow Q) \Leftrightarrow ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$

Now use DeMorgan's law to simplify the following sentences so that only P and Q are negated.

1. $\neg(P \Rightarrow Q)$.
2. $\neg(P \Leftrightarrow Q)$

Negation over Conjunction and Disjunction

Solution:

1.
$$\begin{aligned} \neg(P \Rightarrow Q) &\Leftrightarrow \neg(\neg P \vee Q) \\ &\Leftrightarrow (\neg\neg P \wedge \neg Q) \\ &\Leftrightarrow (P \wedge \neg Q) \end{aligned}$$

Negation over Conjunction and Disjunction

Solution:

1.
$$\begin{aligned} \neg(P \Rightarrow Q) &\Leftrightarrow \neg(\neg P \vee Q) \\ &\Leftrightarrow (\neg\neg P \wedge \neg Q) \\ &\Leftrightarrow (P \wedge \neg Q) \end{aligned}$$

2.
$$\begin{aligned} \neg(P \Leftrightarrow Q) &\Leftrightarrow \neg((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\Leftrightarrow (\neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)) \\ &\Leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)) \end{aligned}$$

Negation

Negation over Quantifiers:

As negation ($\neg$) moves from left to right, it flips universal quantification to existential quantification, and vice versa.

$\exists$:

$\neg(\exists x \in D, P(x) \wedge Q(x)) \Leftrightarrow \forall x \in D, (P(x) \Rightarrow \neg Q(x))$.
In words, "No P is a Q" is equivalent to "Every P is a non-Q."

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In words, “No P is a Q” is equivalent to “Every P is a non-Q.”

$\forall$:

$\neg(\forall x \in D, P(x) \Rightarrow Q(x)) \Leftrightarrow \exists x \in D, (P(x) \wedge \neg Q(x))$.
In words, “Not every P is a Q” is equivalent to “There is some P that is a non-Q.”

Negation over Quantifiers

Exercise:

Simplify the following sentences so that only P and Q are negated.

1. $\neg(\forall x \in D, \exists y \in D, P(x, y))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x, y))$.
3. $\neg(\exists x \in D, \forall y \in D, P(x, y))$.
4. $\neg(\exists x \in D, (P(x) \Rightarrow Q(x)))$.

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow$

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x, y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x, y)))$.

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x,y)))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow$

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x,y)))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, \forall y \in D, \neg P(x,y))$.

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x,y)))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, \forall y \in D, \neg P(x,y))$.
3. $\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow$

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x,y)))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, \forall y \in D, \neg P(x,y))$.
3. $\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \exists y \in D, \neg P(x,y)$.

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x,y)))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, \forall y \in D, \neg P(x,y))$.
3. $\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \exists y \in D, \neg P(x,y)$.
4. $\neg(\exists x \in D, (P(x) \Rightarrow Q(x))) \Leftrightarrow$

Negation over Quantifiers

Solutions:

1. $\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, (\forall y \in D, \neg P(x,y)))$.
2. $\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow (\exists x \in D, \forall y \in D, \neg P(x,y))$.
3. $\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \exists y \in D, \neg P(x,y)$.
4. $\neg(\exists x \in D, (P(x) \Rightarrow Q(x))) \Leftrightarrow (\forall x \in D, (P(x) \wedge \neg Q(x)))$.

Negation

Negation over Quantifiers

Example 1:

$$\neg(\forall x \in D, \exists y \in D, P(x,y)) \Leftrightarrow \exists x \in D, \neg(\exists y \in D, P(x,y))$$
$$\exists x \in D, \neg(\exists y \in D, P(x,y)) \Leftrightarrow \exists x \in D, \forall y \in D, \neg P(x,y)$$

Example 2:

$$\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \neg(\forall y \in D, P(x,y))$$
$$\neg(\exists x \in D, \forall y \in D, P(x,y)) \Leftrightarrow \forall x \in D, \exists y \in D, \neg P(x,y)$$

Example 3:

$$\neg(\exists x \in D, \forall y \in D, (P(x,y) \Rightarrow Q(x,y))) \Leftrightarrow \neg(\exists x \in D, \forall y \in D, (\neg P(x,y) \vee Q(x,y)))$$

CHAPTER 2: LOGICAL NOTATION

LOGICAL ARITHMETIC

Logical Arithmetic

Precedence Rules

Precedence DECREASES from TOP TO BOTTOM:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

Logical Arithmetic

Precedence Rules

Precedence DECREASES from TOP TO BOTTOM:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

Commutative

$P \wedge Q \Leftrightarrow Q \wedge P$ and $P \vee Q \Leftrightarrow Q \vee P$

Logical Arithmetic

Precedence Rules

Precedence DECREASES from TOP TO BOTTOM:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

Commutative

$P \wedge Q \Leftrightarrow Q \wedge P$ and $P \vee Q \Leftrightarrow Q \vee P$

Associative

$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ and $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$