CSC165 Mathematical Expression and Reasoning for Computer Science

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CHAPTER 2: LOGICAL NOTATION

IMPLICATIONS

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Implication

Consider a claim:

"*if* an employee is male, *then* he makes less than 55,000."

This is called an IMPLICATION.

It says that:

- For employees, being male IMPLIES making less than 55,000.
- Or re-expressed as: "Every male employee earns less than 55,000,"
  $\forall x \in E \cap M, L(x)$. (Note: M is a set of male employees.)
- Or, we could have "$M(x)$ implies $L(x)$"
  $\forall x \in E, M(x) \Rightarrow L(x)$

Implication Evaluation

In the implication "if P then Q," we call P the ANTECEDENT (sometimes the ASSUMPTION), and Q the CONSEQUENT (sometimes the CONCLUSION).

Just as with universal quantification:

Disprove

The only way to disprove the implication:

"if P then Q"

is to show an instance where P is true but Q is false.

Prove

If, in every possible instance, we have either not-P or Q ($\neg P \vee Q$), then the implication "if P then Q" is true.

English “if-then” VS. Logic “if-then”

logical implication “if” $\neq$ the English word “if”

English:

“if you eat your vegetables, then you can have dessert.”
means: “otherwise you’ll get no dessert.”

In ordinary English, “if. . . then” means “if and only if. . . then.”

Logic:

“If P then Q.”
means:
“Every P is a Q.”

Example

Claim 1: If an employee is female, then she makes less than 55,000.

Claim discusses three sets,

- E: the set of employees;
- F: the set of female employees;
- and L: the set of employees making less than 55,000.

Claim 1 implicitly invokes universal quantification.

Let's look at that table again.

EMPLOYEE	GENDER	SALARY
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Example

Claim 1: If an employee is female, then she makes less than 55,000.

The Venn diagram Figure 1 indicates the situation corresponding to the table:

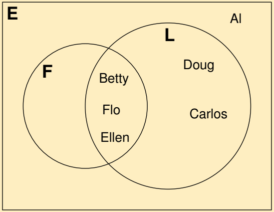

Figure: The only elements of F are also elements of L, so $F \subseteq L$.

We can write implication symbolically as $\Rightarrow$, read "implies." Now "P implies Q" becomes $P \Rightarrow Q$. Claim 1 could now be re-written as

an employee is female $\Rightarrow$ that employee makes less than 55,000.

Implication Evaluation

Disprove

The only way to disprove the implication:

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is to show an instance where P is true but Q is false.

Prove

If, in every possible instance, we have either not-P or Q ($\neg P \vee Q$), then the implication "if P then Q" is true.

Contrapositive

The CONTRAPOSITIVE of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

- the contrapositive of "P implies Q" is "non-Q implies non-P."
- in other words:
  the contrapositive of "all P is/are Q" is "all non-Q is/are non-P."

The contrapositive of the statement has its antecedent and consequent inverted and flipped.

Contrapositive: " an employee is female $\Rightarrow$ that employee makes less than 55,000."

an employee doesn't make less than 55,000 $\Rightarrow$ that employee is not female.

or, given the structure of the domain E of employees:

an employee makes at least 55,000 $\Rightarrow$ that employee is male.

Converse

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.

The converse of "P implies Q" is "Q implies P."

Converse: " an employee is female $\Rightarrow$ that employee makes less than 55,000."

an employee make less than 55,000 $\Rightarrow$ that employee is female.

Exercises 1: Contrapositive & Converse

- if $1 + 1 = 3$, then monkeys can fly.
- You will get a speeding ticket if you drive over 200 km per hour.
- If it snows today, I will ski tomorrow.

Exercise 1: Contrapositive & Converse

- if $1 + 1 = 3$, then monkeys can fly.
  - Contrapositive: if monkeys can not fly, then $1 + 1 \neq 3$.
  - Converse: if monkeys can fly, then $1 + 1 = 3$.
- You will get a speeding ticket if you drive over 200 km per hour.
- If it snows today, I will ski tomorrow.

Exercise 1: Contrapositive & Converse

- if $1 + 1 = 3$, then monkeys can fly.
- You will get a speeding ticket if you drive over 200 km per hour.
  - Contrapositive: if you do not get a speeding ticket, then you will not drive over 200 km per hour.
  - Converse: if you get a speeding ticket, then you will drive over 200 km per hour.
- If it snows today, I will ski tomorrow.

Exercise 1: Contrapositive & Converse

- if $1 + 1 = 3$, then monkeys can fly.
- You will get a speeding ticket if you drive over 200 km per hour.
- If it snows today, I will ski tomorrow.
  - Contrapositive: If I do not ski tomorrow, it does not snow today.
  - Converse: If I ski tomorrow, it snows today.

Quantification and Implication together

Consider the following statement:

(1) " If an employee is male, then that employee makes less than 55,000."

$$\forall x \in E, M(x) \Rightarrow L(x)$$

Implication in English keywords:

- For the antecedent (P) look for: "if," "when," "enough," "sufficient."
- For the consequent (Q) look for: "then," "requires," "must," "need," "necessary," "only if," "when."

In all, check whether the meaning in English matches "either P is false or Q is true" ($\neg P \vee Q$).

Quantification and Implication in English

- If P, [then] Q.
  "If you think I'm lying, then you're a liar!"
- When[ever] P, [then] Q.
  "Whenever I hear that song, I think about ice cream."
- P is sufficient/enough for Q
  "Matching fingerprints and a motive are enough for guilt."
- Can't have P without Q
  "You can't stay enrolled in CSC 165 H without a pulse."
- P requires Q
  "Successful programming requires skill."
- For P to be true, Q must be true / needs to be true / is necessary
  "To pass CSC 165 H, a student needs to get 40% on the final."
- P only if / only when Q
  "I'll go only if you insist."

Exercise 2

Consider the following statement:

(1) "If a program has a syntax error, then the program will not compile."

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.
b. Translate sentence (1) into precise symbolic notation.
c. Give the converse of (1) first in English, then in precise symbolic notation.
d. Give the contrapositive of (1) first in English, then in precise symbolic notation.
e. Give the contrapositive of your answer to [c] in precise symbolic notation and in English words.

Exercise 2

Consider the following statement:

(1) "If a program has a syntax error, then the program will not compile."

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

- Domain: The set of programs, P.
- Let $S(x)$ represent x has a syntax error.
- Let $C(x)$ represent x will compile.

b. Translate sentence (1) into precise symbolic notation.

c. Give the converse of (1) first in English, then in precise symbolic notation.

d. Give the contrapositive of (1) first in English, then in precise symbolic notation.

e. Give the contrapositive of your answer to [c] in precise symbolic notation and in English words.

Exercise 2

Consider the following statement:

(1) "If a program has a syntax error, then the program will not compile."

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

b. Translate sentence (1) into precise symbolic notation.

$$\forall p \in P, S(p) \Rightarrow \neg C(p)$$

c. Give the converse of (1) first in English, then in precise symbolic notation.

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Exercise 2

Consider the following statement:

(1) "If a program has a syntax error, then the program will not compile."

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

b. Translate sentence (1) into precise symbolic notation.

c. Give the converse of (1) first in English, then in precise symbolic notation.

If a program does not compile, then it has a syntax error.

$$\forall p \in P, \neg C(p) \Rightarrow S(p)$$

d. Give the contrapositive of (1) first in English, then in precise symbolic notation.

e. Give the contrapositive of your answer to [c] in precise symbolic notation then in English.

Exercise 2

Consider the following statement:

(1) "If a program has a syntax error, then the program will not compile."

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

b. Translate sentence (1) into precise symbolic notation.

c. Give the converse of (1) first in English, then in precise symbolic notation.

d. Give the contrapositive of (1) first in English, then in precise symbolic notation.

If a program compiles then it does not have a syntax error.

$$\forall p \in P, C(p) \Rightarrow \neg S(p)$$

e. Give the contrapositive of your answer to [c] in precise symbolic notation then in English.

Exercise 2

Consider the following statement:

(1) "If a program has a syntax error, then the program will not compile."

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

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c. Give the converse of (1) first in English, then in precise symbolic notation.

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e. Give the contrapositive of your answer to [c] in precise symbolic notation then in English.

$$\forall p \in P, \neg S(p) \Rightarrow C(p)$$

If a program does not have a syntax error then it will compile.

Exercise 3

Court trial:At a murder trial, four witnesses give the following testimony.

Alice: If either Bob or Carol is innocent, then so am I.

Bob: Alice is guilty, and either Carol or Dan is guilty.

Carol: If Bob is innocent, then Dan is guilty.

Dan: If Bob is guilty, then Carol is innocent; however, Bob is innocent.

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.
b. Translate sentences into precise symbolic notation.
c. Give the converse of (1,3) first in English, then in precise symbolic notation.
d. Give the contrapositive of (1,3) first in English, then in precise symbolic notation.

Exercise 3

Court trial: At a murder trial, four witnesses give the following testimony.

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Dan: If Bob is guilty, then Carol is innocent; however, Bob is innocent.

Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

- A: Alice is innocent
- B: Bob is innocent
- C: Carol is innocent
- D: Dan is innocent

b. Translate sentences into precise symbolic notation.
c. Give the converse of (1,3) first in English, then in precise symbolic notation.
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Questions:

a. Define the domain and predicates necessary to translate the statement into precise symbolic notation.

b. Translate sentences into precise symbolic notation.

Then, we can rewrite the statements symbolically (with the understanding that "guilty $=$ ¬innocent"):

- A: $(B \vee C) \Rightarrow A$
- B: $\neg A \wedge (\neg C \vee \neg D)$
- C: $B \Rightarrow \neg D$
- D: $(\neg B \Rightarrow C) \wedge B$

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- C: $B \Rightarrow \neg D$
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c. Give the converse of (1,3) first in English, then in precise symbolic notation.

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