Chapter 2

Logical Notation

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Announcements

- **Assignment 1**:
  - **Assignment 1** will be posted by Saturday.
  - **Due date**: Jan 30, before midnight on MarkUs.
  - You **won't** be able to log into MarkUs and submit the assignment **before Jan 24**.
  - Assignments may be submitted in groups of up to **two** students. You may choose your group-mate from students in the other section.
  - Submissions must be **typed**. LaTeX is strongly recommended. Some usefule resources and links for LaTeX will be posted on the web page.
  - **Recommendation**: work on all questions individually! Then discuss your answers with your group-mate and prepare the final submission together.

Today's Topics

- **Review: Implication, Contrapositive, Converse**
- **More symbols: Equivalence**
- **Logical Grammar**

Review: Implication

Implication

- An **implication** is a sentence that joins two other sentences and claims that **if** the first part is true **then** the second part is also true.
  - $\mathbf{P} \Rightarrow \mathbf{Q}$: **P** implies **Q**, where **P** and **Q** are **logical sentences**.
- **Implication Symbol**: $\Rightarrow$.

Example

- ***if*** an employee is male, ***then*** he makes less than $55,000.

$$\forall x \in E, M(x) \Rightarrow L(x)$$

Review: Evaluating Implication

- **Important**: $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$

$P \Rightarrow Q$ is **True** if $\neg P$ is **True** or Q is **True**.
$P \Rightarrow Q$ is **False** if $\neg P$ is **False** and Q is **False**.

Evaluating Implication

- To **prove**, verify that **at least one** of $\neg P$ and Q is **True**.
- To **disprove**, show that **both** $\neg P$ and Q are **False**.

Review: Contrapositive and Converse

Contrapositive of an Implication

The contrapositive of an implication $P \Rightarrow Q$ is: $\neg Q \Rightarrow \neg P$

- an employee is female $\Rightarrow$ that employee makes less than \$55,000

Contrapositive:

- an employee doesn't make less than \$55,000 $\Rightarrow$ that employee is not female.

Converse of an Implication

The Converse of an implication $P \Rightarrow Q$ is: $Q \Rightarrow P$

- an employee is female $\Rightarrow$ that employee makes less than \$55,000.

Converse:

- an employee make less than \$55,000 $\Rightarrow$ that employee is female.

Vacuous Truth

Assume for all elements x in a domain U, $T(x)$ is **False**, i.e.

$$\forall x \in U, \neg T(x).$$

Evaluate the following claim:

$$\mathbf{S_1} : \forall x \in U, T(x) \Rightarrow R(x).$$

$\mathbf{S_1}$: is **True**, because

- $T(x) \Rightarrow R(x)$ is equivalent to $\neg T(x) \vee R(x)$.
- $\neg T(x)$ is **True** for all $x \in U$.

$\mathbf{S_1}$ is vacuously true.

Vacuously True Statements

A statement is **vacuously true** if it specifies some properties for all members of an **empty set**!

Vacuous Truth

Vacuously True Statements

A statement is **vacuously true** if it specifies some properties for all members of an **empty set**!

Exercise #1

Evaluate the following claims:

1. $\forall x \in \mathbb{N}, (x < 0) \Rightarrow (x > x + 1)$.
2. $\forall x \in \mathbb{R}, (x^2 + 1 = 0) \Rightarrow (x = 1)$. ($\mathbb{R}$ is the set of real numbers)

Vacuous Truth

Vacuously True Statements

A statement is **vacuously true** if it specifies some properties for all members of an **empty set**!

Exercise #1: solution

Evaluate the following claims:

1. $\forall x \in \mathbb{N}, (x < 0) \Rightarrow (x > x + 1)$.
   Vacuously true, all numbers in $\mathbb{N}$ are greater than or equal to 0.
2. $\forall x \in \mathbb{R}, (x^2 + 1 = 0) \Rightarrow (x = 1)$.
   Vacuously true, there is no number in $\mathbb{R}$ that satisfies $x^2 + 1 = 0$.

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Equivalence

Equivalence

Equivalence

- An **equivalence** is a sentence that joins two other sentences and claims that the first part is true **exactly when** the second part is true.
  - $x^2 + 4x + 4 = 0$ **exactly when** $x = -2$.
- **Equivalence Symbol**: $\Leftrightarrow$.
- **Equivalence in logical notation**: $\mathbf{P} \Leftrightarrow \mathbf{Q}$, where **P** and **Q** are **logical sentences**.
  - $\forall x \in \mathbb{R}, (x^2 + 4x + 4 = 0) \Leftrightarrow (x = -2)$.

Equivalence

Example

Employee	Gender	Salary
Al	male	\$60,000
Betty	female	\$500
Carlos	male	\$40,000
Doug	male	\$30,000
Ellen	female	\$15,000
Flo	female	\$20,000

E: the set of all employees.
$M(x)$: x is male.
$L(x)$: x earns between \$25,000 and \$65,000.

- Every male employee earns between \$25,000 and \$65,000.
  $\forall x \in E, M(x) \Rightarrow L(x)$
- Every employee earning between \$25,000 and \$65,000 is male.
  $\forall x \in E, L(x) \Rightarrow M(x)$
- In this domain, "The employee is male" is equivalent to "the employee earns between \$25,000 and \$65,000".
  $\forall x \in E, M(x) \Leftrightarrow L(x)$

Equivalence

Important Notes about Equivalence

- An equivalence $P \Leftrightarrow Q$ can be written as the conjunction of two implications (we will verify this shortly):

  $$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

- Other ways of expressing equivalence in English:
  - $P \Rightarrow Q$, and **conversely**.
  - P **iff** Q (*iff* is an abbreviation for *if and only if*).
  - P is **necessary** and **sufficient** for Q.
    - sufficient: $P \Rightarrow Q$.
    - necessary: $P \Leftarrow Q$.

Evaluating Equivalence

$P \Leftrightarrow Q$ is **True** if both P and Q are **True** or both P and Q are **False**.
$P \Leftrightarrow Q$ is **False** if P is **True** and Q is **False** or P is **False** and Q is **True**.

Evaluating Equivalence

- To **prove**, verify that the boolean values of P and Q are **equal**.
- To **disprove**, show that the boolean values of P and Q are **not equal**.

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Logical Grammar

Why do we need a grammar?

- There are **seven** different logical symbols: $\forall, \exists, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Combinations of these symbols can form very **complex** statements.
- To write **unambitious** statements we need to impose **conditions** on what expressions are allowed. In other words, we need a **grammar**.

Important Notes

- **Grammatic** (syntactic) correctness has nothing to do with **logical** correctness: a syntacticly correct sentence might be **True** or **False**.
- A syntacticly correct sentence might be **open** or **closed**.

Well-Formed Formulas

A **grammatically correct** sentence is called a
Well-Formed Formula **(wff)**

- Any predicate is a **wff**.
- If P is a **wff**, so is $\neg P$.
- If P and Q are **wffs**, so is $(P \wedge Q)$.
- If P and Q are **wffs**, so is $(P \vee Q)$.
- If P and Q are **wffs**, so is $(P \Rightarrow Q)$.
- If P and Q are **wffs**, so is $(P \Leftrightarrow Q)$.
- If P is a **wff** (possibly open in variable x) and D is a set, then $(\forall x \in D, P)$ is a **wff**.
- If P is a **wff** (possibly open in variable x) and D is a set, then $(\exists x \in D, P)$ is a **wff**.
- Nothing else is a **wff**.

Well-Formed Formulas

- Any predicate is a **wff**. $\rightarrow$ **The base rule**.
- If P is a **wff**, so is $\neg P$.
- If P and Q are **wffs**, so is $(P \wedge Q)$.
- If P and Q are **wffs**, so is $(P \vee Q)$.
- If P and Q are **wffs**, so is $(P \Rightarrow Q)$.
- If P and Q are **wffs**, so is $(P \Leftrightarrow Q)$.
- If P is a **wff** and D is a set, then $(\forall x \in D, P)$ is a **wff**.
- If P is a **wff** and D is a set, then $(\exists x \in D, P)$ is a **wff**.
- Nothing else is a **wff**. $\rightarrow$ **The closure rule**.

Order of Precedence for Logical Symbols

Just like in arithmetic, we want to **avoid** writing expressions with **many parentheses**; too many parentheses make a sentence hard to read:
$\big(\forall x \in D, \big((P(x) \wedge (\neg Q(x))) \Rightarrow R(x)\big)\big)$

Precedence Rules

Precedence **decreases** from **top to bottom**:

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\Rightarrow$
5. $\Leftrightarrow$
6. $\forall$
7. $\exists$

$\forall x \in D, P(x) \wedge \neg Q(x) \Rightarrow R(x)$

Logical Grammar

Order of Precedence:

(1) $\neg$, (2) $\wedge$, (3) $\vee$, (4) $\Rightarrow$, (5) $\Leftrightarrow$, (6) $\forall$, (7) $\exists$

Exercise #2:

Remove as many parentheses as possible:

1. $(\ \forall x \in D, P(x) \wedge (\ (\neg Q(x)) \Rightarrow R(x)))$.
2. $((\forall x \in D, P(x)\ ) \wedge (\ (\neg Q(x)) \Rightarrow R(x)))$.

Logical Grammar

Order of Precedence:

(1) $\neg$, (2) $\wedge$, (3) $\vee$, (4) $\Rightarrow$, (5) $\Leftrightarrow$, (6) $\forall$, (7) $\exists$

Exercise #2: solution

Remove as many parentheses as possible:

1. $(\ \forall x \in D, P(x) \wedge (\ (\neg Q(x)) \Rightarrow R(x)))$.
   $\forall x \in D, P(x) \wedge (\ \neg Q(x) \Rightarrow R(x))$.

2. $((\forall x \in D, P(x)\ ) \wedge (\ (\neg Q(x)) \Rightarrow R(x)))$.
   $(\forall x \in D, P(x)\ ) \wedge (\ \neg Q(x) \Rightarrow R(x))$.

Logical Grammar

Exercise #3:

Are the following sentences **wff**?

1. $R(\forall x \in E, y)$.
2. $\exists x \in E, \exists y \in E$.
3. $\exists x \in E \wedge \exists y \in E$.
4. $R(x, y)$.
5. $\exists x \in E, R(x), S(x)$.
6. $\forall x \in U, \exists y \in E, \forall z \in U, R(x, y) \vee S(x, z)$.

Logical Grammar

Exercise #3: solution

Are the following sentences **wff**?

1. $R(\forall x \in E, y)$. **No**
2. $\exists x \in E, \exists y \in E$. **No**
3. $\exists x \in E \wedge \exists y \in E$. **No**
4. $R(x, y)$. **Yes**
5. $\exists x \in E, R(x), S(x)$. **No**
6. $\forall x \in U, \exists y \in E, \forall z \in U, R(x, y) \vee S(x, z)$. **Yes**