Review

Chapter 4

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Announcements

- **Additional Review Session**: **Tues**, Mar 31, **2-3:45pm** in **MP203**.
- **Additional Instructor Office Hours (Bahar)**:
  - **Mar 30**, 2-4pm, BA3201
  - **Apr 01**, 12-1pm, BA3201
  - **Apr 08**, Time and location TBA.
- **Additional TA Office Hours**:
  - **Mar 30**, 4-6pm, BA3201
  - **Mar 31**, 4-6pm, BA3201
  - More to be announced!

Today's Topics

- **Proving Bounds for Functions**

Review: Asymptotic Notation

$\frac{f(n)}{g(n)} < c$

Reminder: Big-Oh

- $\mathbf{f} \in \mathcal{O}(\mathbf{g})$: g is an **upper bound** of f.
  - For sufficiently large values of n, $g(n)$ multiply by a constant is always **greater than** $f(n)$.
- $\exists \mathbf{c} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow f(n) \leq \mathbf{c}.g(n)$

Reminder: Limits

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$

  $\iff$

  $\forall \varepsilon \in \mathbb{R}^+, \exists \mathbf{n}' \in \mathbb{N}, \forall \mathbf{n} \in \mathbb{N}, \mathbf{n} \geq \mathbf{n}' \Rightarrow \mathbf{L} - \varepsilon < \frac{\mathbf{f(n)}}{\mathbf{g(n)}} < \mathbf{L} + \varepsilon$

$c = L + \varepsilon \Rightarrow c > L$

Proving Bounds for Polynomial Expressions

Proving $\mathcal{O}$ using Limits

Prove $f \in \mathcal{O}(g)$

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$
2. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
3. $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does NOT exist.

Proving Bounds for Polynomial Expressions

Proving $\mathcal{O}$ using Limits

- Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$.
- Prove $f \in \mathcal{O}(g)$:
  1. Choose any value larger than L for c.
  2. Assume $f(n) \leq cg(n)$. Find a value for n such that the inequality holds.
  3. B must be larger than or equal to that value.

Proving Bounds for Polynomial Expressions

Proving $\mathcal{O}$ using Limits

- Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- Prove $f \in \mathcal{O}(g)$:
  1. Assume $c = 1$.
  2. Assume $f(n) \leq cg(n)$. Find a value for n such that the inequality holds.
  3. B must be larger than or equal to that value.

Proving Bounds for Polynomial Expressions

Proving $\mathcal{O}$ using Limits

- Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does NOT exist.
- Prove $f \in \mathcal{O}(g)$:
  1. Find a function $h(n)$ such that $\lim_{n\to\infty} h(n)$ exists, and $\frac{f(n)}{g(n)} \leq h(n)$ for a sufficiently large value n_1 of n.
  2. Choose a value for c such that $c > \lim_{n\to\infty} h(n)$.
  3. Assume $f(n) \leq cg(n)$. Find a value n_2 for n such that the inequality holds.
  4. B must be larger than or equal to $max(n_1, n_2)$.

Review: Asymptotic Notation

$$\frac{f(n)}{g(n)} \geq C$$

Reminder: Big-Ω

- $\mathbf{f} \in \mathbf{\Omega}(\mathbf{g})$: g is an **lower bound** of f.
  - For sufficiently large values of n, $g(n)$ multiply by a constant is always **less than** $f(n)$.
- $\exists \mathbf{c} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow f(n) \geq \mathbf{c}.g(n)$

Reminder: Limits

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$

  $\iff$

  $\forall \varepsilon \in \mathbb{R}^+, \exists \mathbf{n}' \in \mathbb{N}, \forall \mathbf{n} \in \mathbb{N}, \mathbf{n} \geq \mathbf{n}' \Rightarrow \mathbf{L} - \varepsilon < \frac{\mathbf{f(n)}}{\mathbf{g(n)}} < \mathbf{L} + \varepsilon$

$$C = L - \varepsilon \Rightarrow C < L$$

Proving Bounds for Polynomial Expressions

Proving Ω using Limits

Prove $f \in \Omega(g)$

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$
2. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$
3. $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does NOT exist.

Proving Ω using Limits

- Suppose $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$.
- Prove $f \in \Omega(g)$:
  1. Choose any positive value less than L for c.
  2. Assume $f(n) \geq cg(n)$. Find a value for n such that the inequality holds.
  3. B must be larger than or equal to that value.

Proving Bounds for Polynomial Expressions

Proving Ω using Limits

- Suppose $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.
- Prove $f \in \Omega(g)$:
  1. Assume $c = 1$.
  2. Assume $f(n) \geq cg(n)$. Find a value for n such that the inequality holds.
  3. B must be larger than or equal to that value.

Proving Bounds for Polynomial Expressions

Proving Ω using Limits

- Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does NOT exist.
- Prove $f \in \Omega(g)$:
  1. Find a function $h(n)$ such that $\lim_{n\to\infty} h(n)$ exists, and $h(n) \leq \frac{f(n)}{g(n)}$ for a sufficiently large value n_1 of n.
  2. Choose a value for c such that $c < \lim_{n\to\infty} h(n)$.
  3. Assume $f(n) \geq cg(n)$. Find a value n_2 for n such that the inequality holds.
  4. B must be larger than or equal to $max(n_1, n_2)$.

Review: Asymptotic Notation

Big-Theta

- $\mathbf{f} \in \boldsymbol{\Theta}(\mathbf{g})$: g is a **tight bound** of f.
    - For sufficiently large values of n, $g(n)$ is both an **upper bound** and a **lower bound** for $f(n)$.
- $\exists \mathbf{c_1} \in \mathbb{R}^+, \exists \mathbf{c_2} \in \mathbb{R}^+, \exists \mathbf{B} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B} \Rightarrow \mathbf{c_1}.g(n) \leq f(n) \leq \mathbf{c_2}.g(n)$

Proving Big-Theta

- Find c_1 and B_1 such that

$$\exists \mathbf{c_1} \in \mathbb{R}^+, \mathbf{B_1} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B_1} \Rightarrow \mathbf{c_1}.g(n) \leq f(n)$$

- Find c_2 and B_2 such that

$$\exists \mathbf{c_2} \in \mathbb{R}^+, \mathbf{B_2} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq \mathbf{B_2} \Rightarrow f(n) \leq \mathbf{c_2}.g(n)$$

- Then $B = max(B_1, B_2)$.