

Example for Slide #6: Prove that $g(n) = n^4 - 5n^3 - 5n^2 - 5$ is an upper bound for $f(n) = 3n^4 + 3n^3 + 2n^2 + 5$.

Scratch Work:

$$\lim_{n \rightarrow \infty} f(n)/g(n) = 3 \implies L = 3 < c. \text{ Let } c=4.$$

Assume $3n^4 + 3n^3 + 2n^2 + 5 < c(n^4 - 5n^3 - 5n^2 - 5) = 4(n^4 - 5n^3 - 5n^2 - 5)$

$$3n^4 + 3n^3 + 2n^2 + 5 < 4n^4 - 20n^3 - 20n^2 - 20$$

$$25 < n^4 - 23n^3 - 22n^2 = n^2(n^2 - 23n - 22) \implies n^2 - 23n - 22 > 0 \text{ and } n^2 \geq 25$$

$$\implies n^2 - 23n > 22 \implies n(n-23) > 22 \implies n \geq 24 \implies B=24$$

Proof:

Let $c=4, B=25$. # c is a positive real number and B is a natural number
 Assume n is a typical natural number.

Assume $n \geq B$

Then $n(n-23) > 22$ # assume $n \geq B = 24$

Then $n^2 - 23n - 22 > 0$ # algebra

Then $n^2 - 23n - 22 \geq 1$ # there is no natural number between 0 and 1

Then $n^2(n^2 - 23n - 22) \geq 25$ # $n^2 \geq 24^2 > 25$

Then $n^4 - 23n^3 - 22n^2 \geq 25$ # algebra

Then $4n^4 - 23n^3 - 22n^2 \geq 3n^4 + 25$ # add $3n^4$ to both sides

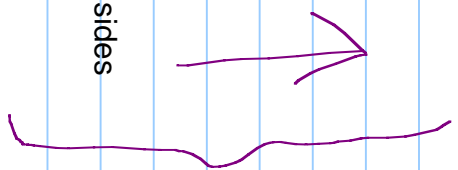
Then $4n^4 - 20n^3 - 20n^2 - 20 \geq 3n^4 + 3n^3 + 2n^2 + 5$ # add $3n^3 + 2n^2 - 20$ to both sides

Then $c(n^4 - 5n^3 - 5n^2 - 5) \geq 3n^4 + 3n^3 + 2n^2 + 5$ # $c=4$

Then $n \geq B \implies c(n^4 - 5n^3 - 5n^2 - 5) \geq 3n^4 + 3n^3 + 2n^2 + 5$ # introduce \implies

Then ... introduce universal and existential quantifiers

going backward through the scratch work



Example for Slide #8: Prove that $g(n)=n^2-2n+1$ is an upper bound for $f(n) = n \cdot \text{floor}(n/2)$.

Scratch Work:

$\text{floor}(n/2) \leq n/2$ for all natural numbers

$n \cdot \text{floor}(n/2) \leq n \cdot n/2 = n^2/2$

Let $f''(n) = n^2/2$ and $h(n) = f''(n)/g(n)$. Then $f(n)/g(n) \leq h(n)$ for all $n \geq 0$ $\implies n_1 = 0$

$\lim_{n \rightarrow \infty} f'(n)/g(n) = 1/2 \implies c = 1$

$n \cdot \text{floor}(n/2) \leq n^2/2$ and assume $n^2/2 \leq c(n^2-2n+1) = n^2-2n+1$

Then $n^2 \leq 2n^2-4n+2$.

Then $0 \leq n^2-4n+2 \quad n \geq 4 \implies n_2 = 4$

$B = \max(n_1, n_2) = 4$

Proof:

Let $c=1, B=4, \# \dots$

(to complete the proof, go through the scratch work backward)

Example for Slide #14: Prove that $g(n) = n^2 - 2n + 1$ is a tight bound for $f(n) = n^2/2$.

Scratch Work:

Lower Bound:

$$\lim_{n \rightarrow \infty} f(n)/g(n) = 1/2 \implies c_1 = 1/3$$

Assume $n^2/2 \geq c_1(n^2/3 - 2n/3 + 1/3)$ where $c_1 = 1/3$

$$n^2/2 \geq n^2/3 - 2n/3 + 1/3$$

$$3n^2 \geq 2n^2 - 4n + 2 \quad \# \text{ mul both sides by 6}$$

$$n^2 + 4n \geq 2 \implies n \geq 1 \implies B_1 = 1$$

Upper Bound:

$$\lim_{n \rightarrow \infty} f(n)/g(n) = 1/2 \implies c_2 = 1$$

Assume $n^2/2 \leq c_2(n^2 - 2n + 1) = n^2 - 2n + 1$ where $c_2 = 1$

$$n^2 \leq 2n^2 - 4n + 2 \quad \# \text{ mul both sides by 2}$$

$$\text{Then } 0 \leq n^2 - 4n + 2 \quad n \geq 4 \implies B_2 = 4$$

$$B = \max(B_1, B_2) = 4$$

Proof:

$$\text{Let } c_1 = 1/3, c_2 = 1, B = 4. \quad \# \dots$$

(to complete the proof, go through the scratch work for both cases backward)