Example for Slide #6: Prove that $g(n) = n^4-5n^3-5n^2-5$ is an upper bound for $f(n) = 3n^4+3n^3+2n^2+5$.

Scratch Work:

$\lim_{n \to \infty} f(n)/g(n)=3$ ===> $L = 3 < c$. Let $c=4$.

Assume $3n^4+3n^3+2n^2+5 =< c\,(n^4-5n^3-5n^2-5) = 4\,(n^4-5n^3-5n^2-5)$

$3n^4+3n^3+2n^2+5 =< 4n^4-20n^3-20n^2-20$

$25 =< n^4-23n^3-22n^2 = n^2(n^2-23n-22)$ ===> $n^2-23n-22 > 0$ and $n^2 >= 25$

===> $n^2-23n > 22$ == > $n(n-23) > 22$ ===> $n >= 24$ ===> $B=24$

going backward through the scratch work

Proof:

Let c=4, B=25. # c is a positive real number and B is a natural number

Assume n is a typical natural number.

- Assume n>= B
  - Then $n(n-23) > 22$ # assume n>= B = 24
  - Then $n^2-23n-22 > 0$ # algebra
  - Then $n^2-23n-22 >= 1$ # there is no natural number between 0 and 1
  - Then $n^2(n^2-23n-22) >= 25$ # $n^2 >= 24^2 > 25$
  - Then $n^4-23n^3-22n^2 >= 25$ # algebra
  - Then $4n^4-23n^3-22n^2 >= 3n^4+25$ # add $3n^4$ to both sides
  - Then $4n^4-20n^3-20n^2 - 20 >= 3n^4+3n^3+2n^2+5$ # add $3n^3+2n^2-20$ to both sides
  - Then $c(n^4-5n^3-5n^2 - 5) >= 3n^4+3n^3+2n^2+5$ # c=4
- Then n>= B => $c(n^4-5n^3-5n^2 - 5) >= 3n^4+3n^3+2n^2+5$ # introduce =>

Then ... introduce universal and exitensial quantifiers

Example for Slide #8: Prove that $g(n)=n^2-2n+1$ is an upper bound for $f(n) = n*floor(n/2)$.

Scratch Work:

$floor(n/2) =< n/2$ for all natural numbers

$n*floor(n/2) =< n*n/2= n^2/2$

Let $f''(n) = n^2/2$ and $h(n) = f''(n)/g(n)$. Then $f(n)/g(n) =< h(n)$ for all $n>=0$ ===> $n_1=0$

$\lim_{n\to infinity} f'(n)/g(n) = 1/2$ ===> $c=1$

$n*floor(n/2) =< n^2/2$ and assume $n^2/2 =< c(n^2-2n+1) = n^2-2n+1$

Then $n^2 =< 2n^2-4n+2$.

Then $0 =< n^2-4n+2$ $n>= 4$ ===> $n_2=4$

$B=max(n_1,n_2)=4$

Proof:

Let c=1, B=4. # ...

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(to complete the proof, go through the scratch work backward)

Example for Slide #14: Prove that $g(n)=n^2-2n+1$ is a tight bound for $f(n) = n^2/2$.

Scratch Work:

Lower Bound:

$\lim_{n \to \text{infinity}} f(n)/g(n) = 1/2$ ===> $c_1=1/3$

Assume $n^2/2 >= c_1(n^2/3-2n/3+1/3)$ where $c_1=1/3$

$n^2/2 >= n^2/3-2n/3+1/3$

$3n^2 >= 2n^2-4n+2$ # mul both sides by 6

$n^2 + 4n >= 2$ ===> $n>= 1$ ===> $B_1=1$

Upper Bound:

$\lim_{n \to \text{infinity}} f(n)/g(n) = 1/2$ ===> $c_2=1$

Assume $n^2/2 =< c_2(n^2-2n+1) = n^2-2n+1$ where $c_2 = 1$

$n^2 =< 2n^2-4n+2$. # mul both sides by 2

Then $0 =< n^2-4n+2$ $n>= 4$ ===> $B_2=4$

$B=\max(B_1,B_2)=4$

Proof:

Let $c_1=1/3$, $c_2 = 1$, $B=4$. # ...

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(to complete the proof, go through the scratch work for both cases backward)