CSC165 Mathematical Expression and Reasoning for Computer Science

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Department of Computer Science
University of Toronto

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General Info

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Lisa: Monday afternoon, Wednesday morning & afternoon sessions.

Bahar: Monday morning, Friday morning & afternoon sessions.

Course Info

- Course web page:

  www.cdf.toronto.edu/~csc165h/winter

- Course info sheet (important):

  www.cdf.toronto.edu/~csc165h/winter/165infosheet.pdf

- Course Notes:

  www.cdf.toronto.edu/~csc165h/winter/165notes.pdf

Chapter 2
Logical Notation

Example

Consider the following table that associates employees w properties:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Claims:

a. Every employee earns less than 70,000.
b. Each employee makes at least 10,000.
c. All female employees make less than 55,000.

How to evaluate these claims are true or false?

Domain

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
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Ellen	female	50,000
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Claims:

a. Every employee earns less than 70,000.
b. Each employee makes at least 10,000.
c. All female employees make less than 55,000.

Analyze and evaluate these claims:

- Claims a,b&c are called statements.
- claims a&b are about the entire database to be considered.

Domain: entire database of all the objects/elements being considered. i.e. here are the six employees.

Sets

Employee	Gender	Salary
Al	male	60,000
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Carlos	male	40,000
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Ellen	female	50,000
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Analyze and evaluate these claims:

- Claims a,b & c are statements.
- These claims are about the entire domain to be considered.

Sets: i.e. define set of employee and denote it as symbol E; the set of female employee as F; the set of male employee as M, the set of employees who earn less than 70,000 as L, etc.

Properties and Sets

- To describe a domain, we write **statements** that specify **properties** of objects within the domain and their **relationships**.
  Recall that we want the **statements** to be in **symbolic** notation.
  To achieve that, **properties** and **relationships** are represented as **sets**.

Example

Emp.	Gender	Supervisor
Al	male	-
Betty	female	Doug
Carlos	male	Ellen
Doug	male	Ellen
Ellen	female	Al
Flo	female	Ellen

- **Property**:
  $\mathbf{M} = \{x \mid x \text{ is } \mathbf{male}\}$.
  $\mathbf{M} = \{Al, Carlos, Doug\}$.
- **Relationship**:
  $\mathbf{S} = \{\langle x, y \rangle \mid x \textbf{ supervises } y\}$.
  $\mathbf{S} = \{\langle Al, Ellen \rangle, \langle Ellen, Carlos \rangle, \langle Ellen, Doug \rangle, \langle Ellen, Flo \rangle, \langle Doug, Betty \rangle\}$.

Predicates

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
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Sets: i.e. define set of employee and denote it as symbol E; the set of female employee as F; the set of male employee as M, the set of employees who earn less than 70,000 as L, etc.

Predicates: M(Betty), L(Carlos), $L(x) : x \in L$

Predicates

Predicate

A unary predicate $L(x)$ is a boolean function returning **True** or **False** such that

$$L(x) = \textbf{True} \text{ if } x \in L.$$

$$L(x) = \textbf{False} \text{ if } x \notin L.$$

An n-ary predicate $L(x_1, ..., x_n)$ is a boolean function returning **True** or **False** such that

$$L(x_1, ..., x_n) = \textbf{True} \text{ if } \langle x_1, ..., x_n \rangle \in L.$$

$$L(x_1, ..., x_n) = \textbf{False} \text{ if } \langle x_1, ..., x_n \rangle \notin L.$$

Predicates

Example

- **Property**:
  $\mathbf{M} = \{Al, Carlos, Doug\}$.
  $M(Al), M(Carlos), M(Doug)$ are **True**.
  $M(Betty), M(Ellen), M(Flo)$ are **False**.
- **Relationship**:
  $\mathbf{S} = \{\langle Al, Ellen\rangle, \langle Ellen, Carlos\rangle, \langle Ellen, Doug\rangle, \langle Ellen, Flo\rangle, \langle Doug, Betty\rangle\}$.
  $S(Al, Ellen), S(Ellen, Carlos), S(Ellen, Doug), S(Ellen, Flo)$
  $S(Doug, Betty)$ are **True**.

Quantifiers

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

Claims:

a. **Every employee earns less than 70,000.**

b. Each employee makes at least 10,000.

c. **All female employees** make less than 55,000.

Sets: employee E; female employee F; male employee M, employees who earn less than 70,000 as L, etc.

Predicate: M(Betty); $L(x) : x \in L$

Quantifier: an expression that indicates the scope. i.e. all employees…, some male employees…

Universal Quantification ∀

- Claim (a): Every employee earns less than 70,000.

  It isn't about one or even several employees; It's about an entire set of employees (employee domain).

- When an statement is about all the objects being considered, the statement is a Universal Quantification.
- Universal quantifier: ∀, 'for all', 'every','any'

  Claim (a): Every employee earns less than 70,000.

  how to use logic notation to represent this?

Universal Quantification ∀

- Claim (a): Every employee earns less than 70,000.

  It isn't about one or even several employees; It's about an entire set of employees (employee domain).

- When an statement is about all the objects being considered, the statement is a Universal Quantification.
- Universal quantifier: ∀ , 'for all', 'every','any'

  Claim (a): Every employee earns less than 70,000.

$$\forall x, x \in E, L(x)$$

Existential Quantification ∃

Another sort of claim appear to be about some individual, un-named, employee.

- Existential quantifier ∃: 'there is', 'some','exsit'
- How to use logic expression to represent:

  Some employee earns less than 70,000 ?

Existential Quantification ∃

Another sort of claim appear to be about some individual, un-named, employee.

- Existential quantifier ∃: 'there is', 'some','exist'
- How to use logic expression to represent:

  Some employee earns less than 70,000 ?

  $\exists x, x \in E, L(x)$

Quick Review

- Domain: all the objects/elements ;
- Sets: define set of employee and denote it as symbol E;
- Predicates: M(Betty), L(Carlos)
- Quantifiers: an expression that indicates the scope of a term:
  - $\forall$ universal quantifier
  - $\exists$ existential quantifier

Exercises

- **All employees earn over 42,000**
- **No male employees earn over 42,000**
- **Some female employee earns over 42,000**

- Express these claims in terms of logic notation & set operations (subsets, intersections, unions, complements, etc.).
- Think of quantification in terms of sets: E is the set of employees, M is the set of male employees, F is the set of female employees, and O is the set of employees earning over 42,000.

Set Theory & Notations

$x \in A$: "x is an element of A."

$\bar{A}$: The set of elements in the domain (universe) that are not in A.

$A \subseteq B$: Every element of A is also an element of B.

$A = B$: A and B contain exactly the same elements, in other words $A \subseteq B$ and $B \subseteq A$.

$A \cup B$: The set of elements that are in either A, or B, or both.

$A \cap B$: The set of elements that are in both A and B.

$A \setminus B$: The set of elements that are in A but not in B (the set difference).

$\emptyset$ or $\{\}$: A set that contains no elements.
For *any* set A, $\emptyset \subseteq A$.

$|A|$: The number of elements in A.

Exercises

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

E: set of employees

M: set of male employees

F: set of female employees

O: set of employees earning over 42,000

All employees earn over 42,000

Logic notations: $\forall x, x \in E, x \in O$

Set operations: $E \subseteq O$

Exercises

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

E: set of employees

M: set of male employees

F: set of female employees

O: set of employees earning over 42,000

No male employees earn over 42,000

Logic notations: $\forall$x, $x \in M, x \in \overline{O}$

Set operations: $M \subseteq \overline{O}$ or $M \cap O = \varnothing$

Exercises

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

E: set of employees

M: set of male employees

F: set of female employees

O: set of employees earning over 42,000

Some female employees earn over 42,000

Logic notations: $\exists x, x \in F, x \in O$

Set operations: $F \cap O \neq \varnothing$ or $F \not\subseteq \overline{O}$

Verify / Disprove ∀

How to verify/disprove a universally-quantified statement:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

a. Every employee earns less than 70,000.
b. All male employees does not earn over 42,000.
c. All female employees earn over 42,000.

Verify / Disprove ∀

How to verify/disprove a universally-quantified statement:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
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Ellen	female	50,000
Flo	female	20,000

a. Every employee earns less than 70,000.
b. All male employees does not earn over 42,000.
c. All female employees earn over 42,000.

To prove:
consider every element in a domain.

To disprove:
exhibit just one counter-example.

Verify / Disprove ∀

A universally-quantified statement of the form

Every P is a Q

needs a single COUNTER-EXAMPLE to disprove, and verification that every element of the domain is an EXAMPLE to prove.

Verify / Disprove ∃

How to verify/disprove existential quantification:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

a. Some employee earns over 80,000.
b. Some male employee earns less than 27,000.
c. Some female employee earns over 42,000.

Verify / Disprove ∃

How to verify/disprove existential quantification:

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

a. Some employee earns over 80,000.
b. Some male employee earns less than 27,000.
c. Some female employee earns over 42,000.

To prove:
exhibit just one example of an element with the property.

To disprove:
consider the entire domain to show that every element is a counter-example.

Anti-symmetry

The anti-symmetry between universal and existential quantification:

	Universal	Existential
Verify (prove)	All elements	one example
Falsify (disprove)	one counter-example	all counter-examples

Conclusion

- Predicates
- Quantifiers:
  - Universal
  - Existential
- Prove / disprove universally-quantified statement and existential quantification