Chapter 2

Logical Notation

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Announcements

- **Tutorials**:
  - Locations and times are posted on the course web page.
  - Tutorial exercises will be posted on the course web page before Monday. Work on the exercise before the tutorial.
  - Each **quiz** covers all topics that you have learned **during the week** prior to the quiz.
- **Office hours**: Friday **12:30-1:30pm** and **3:30-5pm** in **BA4261**.

Today's Topics

- **Evaluating Quantified Statements**
- **Visualization with Venn Diagram**
- **Logical Sentences and Statements**
- **Negation, Conjunction, Disjunction**

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Evaluating Quantified Statements

Review: Sets

Properties and Relationships as Sets

- To describe a domain, we write **statements** that specify **properties** of objects within the domain and their **relationships**.
  One way of writing statements in **symbolic** notation is to treat **properties** and **relationships** as **sets**.

Example

Emp.	Gender	Supervisor
Al	male	-
Betty	female	Doug
Carlos	male	Ellen
Doug	male	Ellen
Ellen	female	Al
Flo	female	Ellen

- **Property**:
  $\mathbf{M} = \{x \mid x \text{ is } \mathbf{male}\}$.
  $\mathbf{M} = \{Al, Carlos, Doug\}$.
- **Relationship**:
  $\mathbf{S} = \{\langle x, y\rangle \mid x \textbf{ supervises } y\}$.
  $\mathbf{S} = \{\langle Al, Ellen\rangle, \langle Ellen, Carlos\rangle, \langle Ellen, Doug\rangle, \langle Ellen, Flo\rangle, \langle Doug, Betty\rangle\}$.

Review: Quantified Statements

- When an statement is about **all** the objects in the domain, the statement is a **Universal Quantification**.
- **Universal quantifier**: $\forall$

- Examples of universally quantified statements in English:
  - All employee makes less than $55,000.
  - Each male employee makes more than $55,000.

- When an statement is about **existence** of one or more elements of a domain with a particular property, the statement is a **Existential Quantification**.
- **Existential quantifier**: $\exists$

- Examples of existentially quantified statements in English:
  - Some employee earns over $65,000.
  - At least one female employee earns less than $65,000.

Evaluating Quantified Statements

- **Prove/disprove** the following universally quantified claims.
  - Every employee makes less than \$55,000.
  - Every female employee makes less than \$50,000.
  - There is no male employee which makes less than \$30,000.

Employee	**Gender**	**Salary**	**Supervisor**
Al	male	\$60,000	-
Betty	female	\$500	Doug
Carlos	male	\$40,000	Ellen
Doug	male	\$30,000	Ellen
Ellen	female	\$50,000	Al
Flo	female	\$20,000	Ellen

Evaluating Universally Quantified Statements

- To **prove**, verify that **all elements** of the domain is an example that satisfies the quantification.
- To **disprove**, give at least **one counter-example** that does not satisfy the quantification.

Evaluating Quantified Statements

- **Prove/disprove** the following existentially quantified claims.
  - Some employee earns less than \$57,000.
  - Some employee earns over \$65,000.
  - Not every female employee earns more than \$10,000.

Employee	Gender	Salary	Supervisor
Al	male	\$60,000	-
Betty	female	\$500	Doug
Carlos	male	\$40,000	Ellen
Doug	male	\$30,000	Ellen
Ellen	female	\$50,000	Al
Flo	female	\$20,000	Ellen

Evaluating Existentially Quantified Statements

- To **prove**, give at least **one example** that satisfies the quantification.
- To **disprove**, verify that **every element** of the domain is a counter-example that does not satisfies the quantification.

Evaluating Quantifiers - Summary

	Universal	Existential
Verify (prove)	All elements	one example
Falsify (disprove)	one counter-example	all counter-examples

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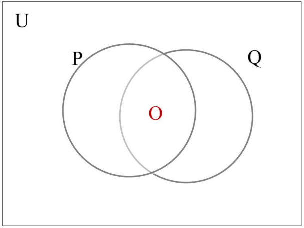
Visualization with Venn Diagram

Visualizing Relationships between Sets

Venn Diagram

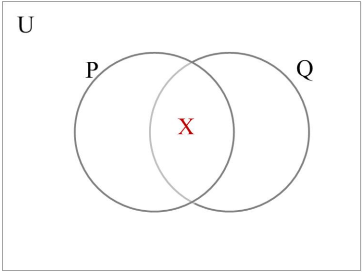
- The **rectangle** represents the **domain**.
- Each **circle** represent **a set** in the domain.
- **O** in a part of a set means that this part must be **occupied**, i.e., there must be some element in there.
- **X** in a part of a set means that this part must be **empty**, i.e., contains no element.

- $P \cap Q \neq \varnothing$



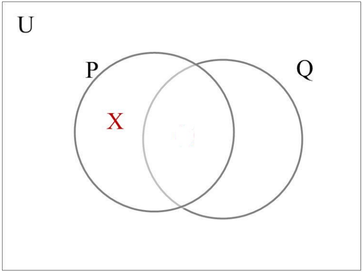
Exercise: Visualizing Relationships between Sets

- **Use Venn Diagram to visualize** $P \cap Q = \varnothing$.



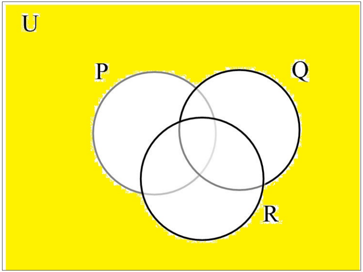
Exercise: Visualizing Relationships between Sets

- **Use Venn Diagram to visualize $P \subseteq Q$.**



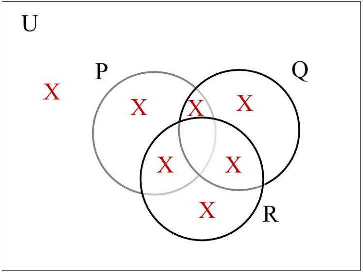
Exercise: Visualizing Relationships between Sets

- **Use Venn Diagram to visualize the region which represents** $\overline{P \cup Q \cup R}$



Exercise: Visualizing Relationships between Sets

- **Use Venn Diagram to visualize $\overline{P \cap Q \cap R} = \varnothing$.**



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Logical Sentences and Statements

Sentences and Statements

What is the difference between following sentences?

- The employee makes less than \$55,000.
- Betty makes less than \$55,000.
- Every employee make less than \$55,000.

Open Sentences vs. Statements

- **Open Sentences** include **unspecified (unquantified) objects**, and therefore **cannot be evaluated**.
- All objects in a **closed sentence** (aka **statement**) are either **specified or quantified**, and therefore a statement can be **evaluated** to True or False.

Open Sentences vs. Statements

Exercise: Is it a statement?

- $L(x)$. **No**
- $\forall x \in E, L(x)$. **Yes**
- $\forall x \in E, S(x, \mathbf{y})$. **No**
- Someone took my pen. **Yes**
- The pen is red. **No**
- Roses are red. **Yes**

Transforming open sentences to statements

- **Specifying** the values of unspecified objects:
  $L(\mathbf{x}) \to L(\mathbf{Carlos})$
- **Quantifying** over unspecified objects:
  $L(\mathbf{x}) \to \forall \mathbf{x}, L(x)$
  $L(\mathbf{x}) \to \exists \mathbf{x}, L(x)$

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Negation, Conjunction, Disjunction

Review: Sets and Predicates

Predicates

An n-ary predicate $L(x_1, ..., x_n)$ is a **boolean function** returning **True** or **False** such that

$L(x_1, ..., x_n) =$ **True** if
$\langle x_1, ..., x_n \rangle$ **satisfy** the property that is denoted by L
$L(x_1, ..., x_n) =$ **False** if
$\langle x_1, ..., x_n \rangle$ **do not satisfy** the property that is denoted by L.

Example

$\mathbf{M} = \{Al, Carlos, Doug\}$.

- $M(Al) =$ **True**, $M(Carlos) =$ **True**, $M(Doug) =$ **True**.
- $M(Betty) =$ **False**, $M(Ellen) =$ **False**, $M(Flo) =$ **False**.

Review: Sets and Predicates

Predicates

An n-ary predicate $L(x_1, ..., x_n)$ is a **boolean function** returning **True** or **False** such that

$L(x_1, ..., x_n) =$ **True** if

$\langle x_1, ..., x_n \rangle$ **satisfy** the property that is denoted by L

$L(x_1, ..., x_n) =$ **False** if

$\langle x_1, ..., x_n \rangle$ **do not satisfy** the property that is denoted by L.

Important Notes about Predicates

- $L(x)$ is **not a set**! In a logical statement, you cannot treat a symbol both as a set and a predicate symbol
  - **Incorrect** use of notation: $\forall x, y \in E$, $x \in L, L(y)$.
  - **Correct** version: $\forall x, y \in E$, $x \in L, y \in L$ or $\forall x, y \in E, L(x) \wedge L(y)$.
- Don't apply **set operations over predicates**!
  $\mathbf{P}(\mathbf{x}) \cap \mathbf{Q}(\mathbf{y})$ makes no sense (why?)
- Don't **nest** predicates!
  $\mathbf{P}(\mathbf{Q}(\mathbf{x}))$ makes no sense (why?)

Negation

Negation Symbol

For the sake of brevity we will write:
$\mathbf{P(x_1, ..., x_n)}$ when $P(x_1, ..., x_n) = \textbf{True}$
$\mathbf{\neg P(x_1, ..., x_n)}$ when $P(x_1, ..., x_n) = \textbf{False}$

- "$\neg$" is called the **negation symbol**.
- $\neg P(x_1, ..., x_n)$ is the negation of predicate $P(x_1, ..., x_n)$.

Example #1

$F(x)$: x feels good.
Translate the following logical sentence to English

- $\neg F(Betty)$: Betty does **not** feel good.
- Can we translate $\neg F(Betty)$ to: Betty feels **bad**?
  - Only if we are given an explicit **assumption**, or we can **formally prove** that all elements in the domain **either feel good or bad**.

Negation

Negation Symbol

For the sake of brevity we will write:

$\mathbf{P(x_1, ..., x_n)}$ when $P(x_1, ..., x_n) = \textbf{True}$

$\neg\mathbf{P(x_1, ..., x_n)}$ when $P(x_1, ..., x_n) = \textbf{False}$

- "$\neg$" is called the **negation symbol**.
- $\neg P(x_1, ..., x_n)$ is the negation of predicate $P(x_1, ..., x_n)$.

Example #2

$M(x)$: x is male.

Translate the following logical sentence to English

- $\neg M(Betty)$: Betty is **not male**.
- Can we translate $\neg M(Betty)$ to: Betty **is female**?
  - Only if we are given an explicit **assumption**, or we can **formally prove** that all elements in the domain are **either male or female**.

Negation

Negation Symbol

For the sake of brevity we will write:
$\mathbf{P(x_1, ..., x_n)}$ when $P(x_1, ..., x_n) =$ **True**
$\mathbf{\neg P(x_1, ..., x_n)}$ when $P(x_1, ..., x_n) =$ **False**

- "$\neg$" is called the **negation symbol**.
- $\neg P(x_1, ..., x_n)$ is the negation of predicate $P(x_1, ..., x_n)$.

Example #3

$L(x)$: x earns less than \$55,000.
Translate the following logical sentence to English

- $L(x)$: x earns less than \$55,000.
  $\neg L(Al)$: Al does not earn less than \$55,000.
- Can we translate $\neg L(Al)$ to: Al earns more than or equal to \$55,000?
  - **Yes**, because we have the following **mathematical fact** about numbers: For two numbers n and m, either $\mathbf{n = m}$ or $\mathbf{n < m}$ or $\mathbf{n > m}$.

Conjunction (Logical AND)

Conjunctive Sentences

- A **conjunction** is a sentence that joins two other sentences and claims that **both** of the original sentences are true.
  - Al makes more than \$25,000 **and** less than \$75,000.
- **Conjunct Symbol**: $\wedge$
- **Conjunction in logical notation**: $\mathbf{P} \wedge \mathbf{Q}$, where **P** and **Q** are **logical sentences**.

$L(x)$: x earns less than \$75,000.
$K(x)$: x earns more than \$25,000.

- Al makes more than \$25,000 and less than \$75,000.
  $K(Al) \wedge L(AL)$.
- All employees make more than \$25,000 and less than \$75,000.
  $\forall x \in E, K(x) \wedge L(x)$.

Evaluating Conjunctions

$P \wedge Q$ is **True** if P is **True** and Q is **True**.
$P \wedge Q$ is **False** if P is **False** or Q is **False**.

Evaluating Conjunctions

- To **prove**, verify that **both** P and Q are **True**.
- To **disprove**, show that **at least one** of P and Q is **False**.

Disjunction (Logical OR)

Disjunctive Sentences

- A **disjunction** is a sentence that joins two other sentences and claims that **at least on** of the original sentences are true.
  - The employee is female **or** makes less than \$75,000.
- **Disjunct Symbol**: $\vee$
- **Disjunction in logical notation**: $\mathbf{P} \vee \mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are **logical sentences**.

$L(x)$: x earns less than \$75,000.
$F(x)$: x is female.

- The employee is female or makes less than \$75,000.
  $x \in E, F(x) \vee L(x)$.
- All employees are female or make less than \$75,000.
  $\forall x \in E, F(x) \vee L(x)$.

Evaluating Disjunctions

$P \vee Q$ is **True** if P is **True** or Q is **True**.
$P \vee Q$ is **False** if P is **False** and Q is **False**.

Evaluating Disjunctions

- To **prove**, verify that **at least one** of P and Q is **True**.
- To **disprove**, show that **both** P and Q are **False**.

After this lecture ...

- You should be able to understand the following jokes:

Three logicians walk into a bar.
The **bartender** asks: Do all of you want a drink?
The **first logician** says: I don't know.
The **second logician** says: I don't know.
The **third logician** says: Yes!

A **logician's wife** is having a **baby**.
The doctor immediately hands the **newborn** to the **dad**.
His **wife** asks impatiently: So, is it a boy or a girl?
The **logician** replies: Yes.
(Well, it seems that the logician has made an assumption, right?)