CSC165 Mathematical Expression and Reasoning for Computer Science

Chapter 4: Algorithm Analysis and Asymptotic Notation

Lisa Yan

Department of Computer Science
University of Toronto

March 23, 2015

Announcements

- Tutorial session: Tuesday (MP203) & Thursday (MP103)
- Evaluation Scheme Revision Voting Date:
  - LEC0101: March 27 11am
  - LEC0201: March 25 2pm

Asymptotic Notation

$\mathcal{O}$ **Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$ (*i.e.*, any function mapping naturals to nonnegative reals), let

$$\mathcal{O}(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \leqslant cf(n)\}.$$

"g grows no faster than f". ($\mathbb{R}^{+}$: the set of positive real numbers.)

Ω **Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Omega(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \geqslant cf(n)\}.$$

"g grows at least as fast as f".

Θ **Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Theta(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c_1 \in \mathbb{R}^{+}, \exists c_2 \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow c_1 f(n) \leqslant g(n) \leqslant c_2 f(n)\}.$$

"g grows at the same rate as f".

Calculus: *Limit*

Recall the following definition, for all $L \in \mathbb{R}^{\geqslant 0}$:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow \frac{f(n)}{g(n)} > \varepsilon$$

Induction

Suppose $P(n)$ is some predicate of the natural numbers, and:

$$(*) \quad P(0) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)).$$

$(*)$ implies $P(n)$ for any natural number n.

You should be able to show that $(*)$ implies $P(0)$, $P(1)$, $P(2)$, in fact $P(n)$ where n is any natural number you have the patience to follow the chain of results to obtain.

PSI

This is called the *Principle of Simple Induction*. (It isn't proved, it is an axiom that we assume to be true.)

Prove: $\forall n, P(n) : 2^n \geqslant 2n$

Prove by Induction: using the Principle of Simple Induction.

Prove $P(0)$: $P(0)$ states that $2^0 = 1 \geqslant 2(0) = 0$, which is true.

Prove $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$:

Assume $n \in \mathbb{N}$. # arbitrary natural number

Assume $P(n)$, that is $2^n \geqslant 2n$. # antecedent

Then $n = 0 \vee n > 0$. # natural numbers are non-negative

Case 1 (assume $n = 0$): Then $2^{n+1} = 2^1 = 2 \geqslant 2(n+1) = 2$.

Case 2 (assume $n > 0$): Then $n \geqslant 1$.
Then $2n \geqslant 2$.
Then $2^{n+1} = 2^n + 2^n \geqslant 2n + 2n \geqslant 2n + 2 = 2(n+1)$.

Then $2^{n+1} \geqslant 2(n+1)$, which is $P(n+1)$. # true in both possible cases

Then $P(n) \Rightarrow P(n+1)$. # introduce $\Rightarrow$

Then $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$. # introduce $\forall$

Now conclude, by the PSI, $\forall n \in \mathbb{N}, P(n)$, that is $2^n \geqslant 2n$.

$P(n) : 2^n > n^2$?

For example, 2^n grows much more quickly than n^2, but 2^3 is not larger than 3^2. Choose n big enough, though, and it is true that:

$$P(n) : 2^n > n^2.$$

You can't prove this for all n, when it is false for $n = 2, n = 3$, and $n = 4$, so you'll need to restrict the domain and prove that for all natural numbers greater than 4, $P(n)$ is true.

What happens to induction for predicates that are true for all natural numbers after a certain point, but untrue for the first few natural numbers?

Let's consider three ways to restrict the natural numbers to just those greater than 4, and then use induction.

Restriction

Restrict by set difference: One way to restrict the domain is by set difference:

$$\forall n \in \mathbb{N} \setminus \{0, 1, 2, 3, 4\}, P(n)$$

Again, we'll need to prove $P(5)$, and then that $\forall n \in \mathbb{N} \setminus \{0, 1, 2, 3, 4\}, P(n) \Rightarrow P(n+1)$.

Restrict by translation: We can also restrict the domain by translating our predicate, by letting $Q(n) = P(n+5)$, that is:

$$Q(n) : 2^{n+5} > (n+5)^2$$

Now our task is to prove $Q(0)$ is true and that for all $n \in \mathbb{N}$, $Q(n) \Rightarrow Q(n+1)$. This is simple induction.

Restrict using implication: Another method of restriction uses implication to restrict the domain where we claim $P(n)$ is true — in the same way as for sentences:

$$\forall n \in \mathbb{N}, n \geqslant 5 \Rightarrow P(n).$$

Prove

After all that work, it turns out that we need prove just two things:

1. $P(5)$
2. $\forall n \in \mathbb{N}$, If $n > 5$, then $P(n) \Rightarrow P(n+1)$.

This is the same as before, except now our base case is $P(5)$ rather than $P(0)$, and we get to use the fact that $n \geqslant 5$ in our induction step (if we need it).

Basic steps for simple induction:

prove the base case (which may now be greater than 0)
prove the induction step