CSC165 Mathematical Expression and Reasoning for Computer Science

Chapter 4: Algorithm Analysis and Asymptotic Notation

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Asymptotic notation

- $\mathcal{O}$
- Ω
- Θ

Big-O Notation

Here is a precise definition of "The set of functions that are eventually no more than f, to within a constant factor":

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$ (*i.e.*, any function mapping naturals to nonnegative reals), let

$$\mathcal{O}(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \leqslant cf(n)\}.$$

$g \in \mathcal{O}(f)$ means that "g grows no faster than f".
Equivalently, "f is an upper bound for g".

$\mathbb{R}^{+}$: the set of positive real numbers

Ω Notation

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Omega(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \geqslant cf(n)\}.$$

"$g \in \Omega(f)$" expresses the concept that "g grows at least as fast as f";
f is a lower bound on g.

Θ Notation

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Theta(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow c_1 f(n) \leqslant g(n) \leqslant c_2 f(n)\}.$$

"$g \in \Theta(f)$" expresses the concept that "g grows at the same rate as f".
f is a tight bound for g, or f is both an upper bound and a lower bound on g.

Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6$ is in $\mathcal{O}(n^2 - 4n^5 + 6n^8)$

We begin with ...

Let $c' =$ ____. Then $c' \in \mathbb{R}^+$.
Let $B' =$ ____. Then $B' \in \mathbb{N}$.
Assume $n \in \mathbb{N}$ and $n \geqslant B'$. # arbitrary natural number and antecedent
    Then $2n^3 - 5n^4 + 7n^6 \leqslant \ldots \leqslant c'(n^2 - 4n^5 + 6n^8)$.
Then $\forall n \in \mathbb{N}, n \geqslant B' \Rightarrow 2n^3 - 5n^4 + 7n^6 \leqslant c'(n^2 - 4n^5 + 6n^8)$. # introduce $\Rightarrow$ and $\forall$
Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 2n^3 - 5n^4 + 7n^6 \leqslant c(n^2 - 4n^5 + 6n^8)$. # introduce $\exists$

Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6 \in \mathcal{O}(n^2 - 4n^5 + 6n^8)$

To fill in the ...

we try to form a chain of inequalities, working from both ends, simplifying the expressions:

$$
\begin{aligned}
2n^3 - 5n^4 + 7n^6 &\leqslant 2n^3 + 7n^6 \quad \text{(drop } -5n^4\text{)} \\
&\leqslant 2n^6 + 7n^6 \quad \text{(increase } n^3 \text{ to } n^6\text{)} \\
&= 9n^6 \leqslant 9n^8 \qquad \text{(simpler to compare)} \\
&= 2(9/2)n^8 \quad \text{(choose } c' = 9/2\text{)} \\
&= 2cn^8 \\
&= c'(-4n^8 + 6n^8) \quad \text{(bottom up: decrease } -4n^5 \text{ to } -4n^8\text{)} \\
&\leqslant c'(-4n^5 + 6n^8) \quad \text{(bottom up: drop } n^2\text{)} \\
&\leqslant c'(n^2 - 4n^5 + 6n^8)
\end{aligned}
$$

We never needed to restrict n for $n \in \mathbb{N}$ ($n \geqslant 0$), so we can fill in $c' = 9/2$, $B' = 0$, and complete the proof.

Exercises

Here are some general results that we now have the tools to prove.

- $3n^2 + 2n \in \mathcal{O}(n^2)$.
- $n^3 \notin \mathcal{O}(3n^2)$.
- $2^n \notin \mathcal{O}(n^2)$.
- $n^2 + n \in \Omega(15n^2 + 3)$.

Calculus

Intuitively, big-Oh notation expresses something about how two functions compare as n tends toward infinity. But we know of another mathematical notion that captures a similar (though not identical) idea: the concept of *limit*.

precisely, recall the following definition, for all $L \in \mathbb{R}^{\geqslant 0}$:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = L \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon$$

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and the following special case:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow \frac{f(n)}{g(n)} > \varepsilon$$

Prove: $f(n) \in \mathcal{O}(g(n))$

Suppose that $\lim_{n\to\infty} f(n)/g(n) = L$. Intuitively, this tells us that $f(n)/g(n) \approx L$, for n "large enough."

In that case, $f(n) \approx Lg(n)$ for n large enough, so we should be able to prove that $f \in \mathcal{O}(g)$:

Assume $\lim_{n\to\infty} f(n)/g(n) = L$.
  Then $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow L - 0.9 < f(n)/g(n) < L + 0.9$. # definition of limit for $\varepsilon = 0.9$
  Then $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow f(n) < (L + 0.9)g(n) \leqslant (L + 1)g(n)$.
  Then $f \in \mathcal{O}(g)$. # definition of $\mathcal{O}$, with $B = n_0$ and $c = L + 1$
Hence, $\lim_{n\to\infty} f(n)/g(n) = L \Rightarrow f \in \mathcal{O}(g)$.

Prove: $g(n) \notin \mathcal{O}(f(n))$

Recall that $g(n) = 2^n$ and $f(n) = n$. We rely on the fact that $\lim_{n\to\infty} 2^n/n = \infty$.[1]

Assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$. # arbitrary values

Then $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow 2^n/n > c$. # definition of $\lim_{n\to\infty} 2^n/n = \infty$ with $\varepsilon = c$

Let n_0 be such that $\forall n \in \mathbb{N}, n \geqslant n_0 \Rightarrow 2^n/n > c$, and $n' = \max(B, n_0)$.

Then $n' \in \mathbb{N}$.

Then $n' \geqslant B$. # by definition of $\max$

Then $2^{n'} > cn'$ because $2^{n'}/n' > c$. # by the first line above, since $n' \geqslant n_0$

Then $n' \geqslant B \wedge g(n') \geqslant cf(n')$. # introduce $\wedge$

Then $\exists n \in \mathbb{N}, n \geqslant B \wedge g(n) \geqslant cf(n)$. # introduce $\exists$

Then $\forall c \in \mathbb{R}, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geqslant B \wedge g(n) > cf(n)$. # introduce $\forall$

[1] Applying l'Hôpital's Rule, $\lim_{n\to\infty} \frac{2^n}{n} = \lim_{n\to\infty} \frac{\ln(2)\cdot 2^n}{1} = \infty$.

Exercises

Here are some general results that we now have the tools to prove.

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- $n^2 + n \in \Omega(15n^2 + 3)$.

Some Theorems

Here are some general results that we now have the tools to prove.

- $(f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$.
  Intuition: If f grows no faster than g, and g grows no faster than h, then f must grow no faster than h.
- $g \in \Omega(f) \Leftrightarrow f \in \mathcal{O}(g)$.
  Intuition: if f grows no faster than g, then g grows no slower than f.
- $g \in \Theta(f) \Leftrightarrow g \in \mathcal{O}(f) \wedge g \in \Omega(f)$.
  Intuition: g grows at the same rate as f. f is both an upper bound and a lower bound on g.

Theorem 1

Theorem

For any functions $f, g, h : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, we have $(f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$.

PROOF:

Assume $f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)$.
- So $f \in \mathcal{O}(g)$.
- So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n > B \Rightarrow f(n) \leqslant cg(n)$. # by def'n of $f \in \mathcal{O}(g)$
- Let $c_g \in \mathbb{R}^+, B_g \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geqslant B_g \Rightarrow f(n) \leqslant c_g g(n)$.
- So $g \in \mathcal{O}(h)$.
- So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \leqslant ch(n)$. # by def'n of $g \in \mathcal{O}(h)$
- Let $c_h \in \mathbb{R}^+, B_h \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geqslant B_h \Rightarrow g(n) \leqslant c_h h(n)$.
- Let $c' = c_g c_h$. Let $B' = \max(B_g, B_h)$.
- Then, $c' \in \mathbb{R}^+$ (because $c_g, c_h \in \mathbb{R}^+$) and $B' \in \mathbb{N}$ (because $B_g, B_h \in \mathbb{N}$).
- Assume $n \in \mathbb{N}$ and $n \geqslant B'$.
  - Then $n \geqslant B_h$ (by definition of $\max$), so $g(n) \leqslant c_h h(n)$.
  - Then $n \geqslant B_g$ (by definition of $\max$), so $f(n) \leqslant c_g g(n) \leqslant c_g c_h h(n)$.
  - So $f(n) \leqslant c' h(n)$.
- Hence, $\forall n \in \mathbb{N}, n \geqslant B' \Rightarrow f(n) \leqslant c' h(n)$.
- Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow f(n) \leqslant ch(n)$.
- So $f \in \mathcal{O}(g)$, by definition.

So $(f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

Theorem 2

Theorem

For any functions $f, g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, we have $g \in \Omega(f) \Leftrightarrow f \in \mathcal{O}(g)$.

PROOF:

$g \in \Omega(f)$

$\iff \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \geqslant cf(n)$(by definition)

$\iff \exists c' \in \mathbb{R}^+, \exists B' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B' \Rightarrow f(n) \leqslant c'g(n)$

(letting $c' = 1/c$ and $B' = B$)

$\iff f \in \mathcal{O}(g)$(by definition)

Theorem 3

Theorem

For any functions $f, g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, we have $g \in \Theta(f) \Leftrightarrow g \in \mathcal{O}(f) \wedge g \in \Omega(f)$.

PROOF:

$g \in \Theta(f)$

$\Leftrightarrow$ (by definition)

$\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow c_1 f(n) \leqslant g(n) \leqslant c_2 f(n).$

$\Leftrightarrow$ (combined inequality, and $B = \max(B_1, B_2)$)

$\left(\exists c_1 \in \mathbb{R}^+, \exists B_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B_1 \Rightarrow g(n) \geqslant c_1 f(n)\right) \wedge$

$\left(\exists c_2 \in \mathbb{R}^+, \exists B_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B_2 \Rightarrow g(n) \leqslant c_2 f(n)\right)$

$\Leftrightarrow$ (by definition)

$g \in \Omega(f) \wedge g \in \mathcal{O}(f)$

Theorem 4

Corollary: For any functions $f, g : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, we have $g \in \Theta(f) \Leftrightarrow f \in \Theta(g)$.

Proof:

$$
\begin{aligned}
& g \in \Theta(f) \\
\iff & g \in \mathcal{O}(f) \wedge g \in \Omega(f) && \text{(by 3)} \\
\iff & g \in \mathcal{O}(f) \wedge f \in \mathcal{O}(g) && \text{(by 2)} \\
\iff & f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(f) && \text{(by commutativity of } \wedge) \\
\iff & f \in \mathcal{O}(g) \wedge f \in \Omega(g) && \text{(by 2)} \\
\iff & f \in \Theta(g) && \text{(by 3)}
\end{aligned}
$$