Asymptotic notation

- $\mathcal{O}$
- Ω
- Θ

Big-O Notation

Here is a precise definition of "The set of functions that are eventually no more than f, to within a constant factor":

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$ (*i.e.*, any function mapping naturals to nonnegative reals), let

$$\mathcal{O}(f) = \left\{ g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \leqslant cf(n) \right\}.$$

$g \in \mathcal{O}(f)$ means that "g grows no faster than f".
Equivalently, "f is an upper bound for g".

$\mathbb{R}^{+}$: the set of positive real numbers

$\mathcal{O}$ Proof Example

Suppose: $g(n) = 3n^2 + 2$ and $f(n) = n^2$

Then $g \in \mathcal{O}(f)$.
To be more precise, we need to prove the statement

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 3n^2 + 2 \leqslant cn^2.$$

Find some c and B that "work" in order to prove the theorem.

Prove: $\exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 3n^2 + 2 \leqslant cn^2$

Idea:

Finding c means finding a factor that will scale n^2 up to the size of $3n^2 + 2$. Setting $c = 3$ almost works, but there's that annoying additional term 2. Certainly $3n^2 + 2 < 4n^2$ so long as $n \geqslant 2$, since $n \geqslant 2 \Rightarrow n^2 > 2$. So pick $c = 4$ and $B = 2$

Prove: $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 3n^2 + 2 \leqslant cn^2$

Idea:

Finding c means finding a factor that will scale n^2 up to the size of $3n^2 + 2$. Setting $c = 3$ almost works, but there's that annoying additional term 2. Certainly $3n^2 + 2 < 4n^2$ so long as $n \geqslant 2$, since $n \geqslant 2 \Rightarrow n^2 > 2$. So pick $c = 4$ and $B = 2$

Let $c' = 4$ and $B' = 2$.
Then $c' \in \mathbb{R}^+$ and $B' \in \mathbb{N}$.
Assume $n \in \mathbb{N}$ and $n \geqslant B'$. # direct proof for an arbitrary natural number
  Then $n^2 \geqslant B'^2 = 4$. # squaring is monotonic on natural numbers
  Then $n^2 \geqslant 2$.
  Then $3n^2 + n^2 \geqslant 3n^2 + 2$. # adding $3n^2$ to both sides of the inequality
  Then $3n^2 + 2 \leqslant 4n^2 = c'n^2$ # re-write
Then $\forall n \in \mathbb{N}, n \geqslant B' \Rightarrow 3n^2 + 2 \leqslant c'n^2$ # introduce $\forall$ and $\Rightarrow$
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 3n^2 + 2 \leqslant cn^2$. # introduce $\exists$ (twice)

So, by definition, $g \in \mathcal{O}(f)$.

Ω, Θ Notation

By analogy with $\mathcal{O}(f)$, consider two other definitions:

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Omega(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \geqslant cf(n)\}.$$

"$g \in \Omega(f)$" expresses the concept that "g grows at least as fast as f";
f is a lower bound on g.

Ω, Θ Notation

By analogy with $\mathcal{O}(f)$, consider two other definitions:

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Omega(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow g(n) \geqslant cf(n)\}.$$

"$g \in \Omega(f)$" expresses the concept that "g grows at least as fast as f";
f is a lower bound on g.

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$, let

$$\Theta(f) = \{g : \mathbb{N} \to \mathbb{R}^{\geqslant 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow c_1 f(n) \leqslant g(n) \leqslant c_2 f(n)\}.$$

"$g \in \Theta(f)$" expresses the concept that "g grows at the same rate as f".
f is a tight bound for g, or f is both an upper bound and a lower bound on g.

Next week: more complex examples

Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6$ is in $\mathcal{O}(n^2 - 4n^5 + 6n^8)$

We begin with ...

Let $c' =$ ____. Then $c' \in \mathbb{R}^+$.
Let $B' =$ ____. Then $B' \in \mathbb{N}$.
Assume $n \in \mathbb{N}$ and $n \geqslant B'$. # arbitrary natural number and antecedent
Then $2n^3 - 5n^4 + 7n^6 \leqslant \ldots \leqslant c'(n^2 - 4n^5 + 6n^8)$.
Then $\forall n \in \mathbb{N}, n \geqslant B' \Rightarrow 2n^3 - 5n^4 + 7n^6 \leqslant c'(n^2 - 4n^5 + 6n^8)$. # introduce $\Rightarrow$ and $\forall$
Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 2n^3 - 5n^4 + 7n^6 \leqslant c(n^2 - 4n^5 + 6n^8)$. # introduce $\exists$

Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6 \in \mathcal{O}(n^2 - 4n^5 + 6n^8)$

To fill in the . . .

we try to form a chain of inequalities, working from both ends, simplifying the expressions:

$$
\begin{aligned}
2n^3 - 5n^4 + 7n^6 &\leqslant 2n^3 + 7n^6 \quad \text{(drop } -5n^4\text{)} \\
&\leqslant 2n^6 + 7n^6 \quad \text{(increase } n^3 \text{ to } n^6\text{)} \\
&= 9n^6 \leqslant 9n^8 \qquad \text{(simpler to compare)} \\
&= 2(9/2)n^8 \quad \text{(choose } c' = 9/2\text{)} \\
&= 2cn^8 \\
&= c'(-4n^8 + 6n^8) \quad \text{(bottom up: decrease } -4n^5 \text{ to } -4n^8\text{)} \\
&\leqslant c'(-4n^5 + 6n^8) \quad \text{(bottom up: drop } n^2\text{)} \\
&\leqslant c'(n^2 - 4n^5 + 6n^8)
\end{aligned}
$$

We never needed to restrict n for $n \in \mathbb{N}$ ($n \geqslant 0$), so we can fill in $c' = 9/2$, $B' = 0$, and complete the proof.