CHAPTER 4

ALGORITHM ANALYSIS AND ASYMPTOTIC NOTATION

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Computer scientists talk like…

"The worst-case runtime of bubble-sort is in $\boxed{O(n^2)}$."

"I can sort it in $\boxed{n \log n \text{ time}}$."

"That's too slow, make it $\boxed{\text{linear-time}}$."

"That problem cannot be solved in $\boxed{\text{polynomial time}}$."

Sorting Algorithms Comparison

- Bubble sort
- Merge sort

See demo at: http://www.sorting-algorithms.com/

Observations:

✧ **merge** is faster than **bubble**

✧ with larger input size, the advantage of **merge** over **bubble** becomes larger

Runtime Observation

Algorithm/Size	20 (s)	40 (s)
Bubble	8.6	38.0
Merge	5.0	11.2

When input size grows from 20 to 40…

- “runtime” of merge: roughly doubled
- “runtime” of bubble: roughly quadrupled

Runtime means?

- ✧ It does **NOT** mean how many **seconds** spent on running the algorithm.
- ✧ It means the **number of steps** taken by the algorithm.

Thus, the runtime is **independent** from the **hardware** where you run the algorithm; but, only depends on the algorithm itself.

You can run **bubble** on a super-computer, and run **merge** on a mechanical watch! That has nothing to do with the fact that **merge** is a faster sorting algorithm than **bubble**.

Runtime Description

Algorithm/Size	20 (steps)	40 (steps)
Bubble	200	800
Merge	120	295

The runtime described in number of steps, as a function of ***n*** (size of input):

- Bubble: could be **$0.5n^2$** (steps)
- Merge: could be **n log n** (steps)

But, we don't really care about the number of steps...

Size n & step numbers?

We **don't care** about the **absolute** number of steps, but

we care about:

when input size **doubles**, the runtime **quadruples.**

In other words, $0.5n^2$ and $700n^2$ are no different!

What we really care is that:

- how the number of steps grows as the size of input increases.

Constant factors do NOT matter!

Constant factor, steps grow?

$$T_1(n) = 0.5\,n^2 \qquad T_2(n) = 700\,n^2$$

$$\frac{T_1(2n)}{T_1(n)} = \frac{0.5\,(2n)^2}{0.5\,n^2} = \frac{2n^2}{0.5n^2} = 4$$

$$\frac{T_2(2n)}{T_2(n)} = \frac{700\,(2n)^2}{700\,n^2} = \frac{2800\,n^2}{700\,n^2} = 4$$

Constant factor does not matter, when it comes to growth!

Large input sizes

We care about algorithm design when the input size **n** is very large.

- ✧ n^2 and n^2+n+2 are no different, because when n is really large, $n+2$ is negligible compared to n^2
- ✧ **Only the highest-order term matters!**

Low-order terms

Low-order terms do not matter!

$$T_1(n) = n^2 \qquad T_2(n) = n^2 + n + 2$$

$$T_1(10000) = 100,000,000$$

$$T_2(10000) = 100,010,002$$

$$\text{difference} \approx 0.01\%$$

Summary of Runtime

Runtime evaluation:

➔ we count the number of steps

➔ constant factors don't matter

➔ only the highest-order term matters

Thus, the followings functions are of the same class:

$$n^2 \qquad 2n^2 + 3n \qquad \frac{n^2}{165} + 1130n + 3.14159$$

We call this: $O(n^2)$

Big-O Notation

$O(n^2)$ is an asymptotic notation

O(f(n)) is the asymptotic upper-bound, which means that a set of functions grow no faster than f(n).

For example, when we say: $5n^2 + 3n + 1$ is in $O(n^2)$

It means that:

$5n^2+3n+1$ grows no faster than n^2, asymptotically

Asymptotic Notations

More notations to be introduced later:

- $O(f(n))$: the asymptotic upper-bound
- $\Omega(f(n))$: the asymptotic lower-bound
- $\Theta(f(n))$: the asymptotic tight-bound

Precise definitions of O, Ω, and Θ to be given in next class

Asymptotic notations: abstraction

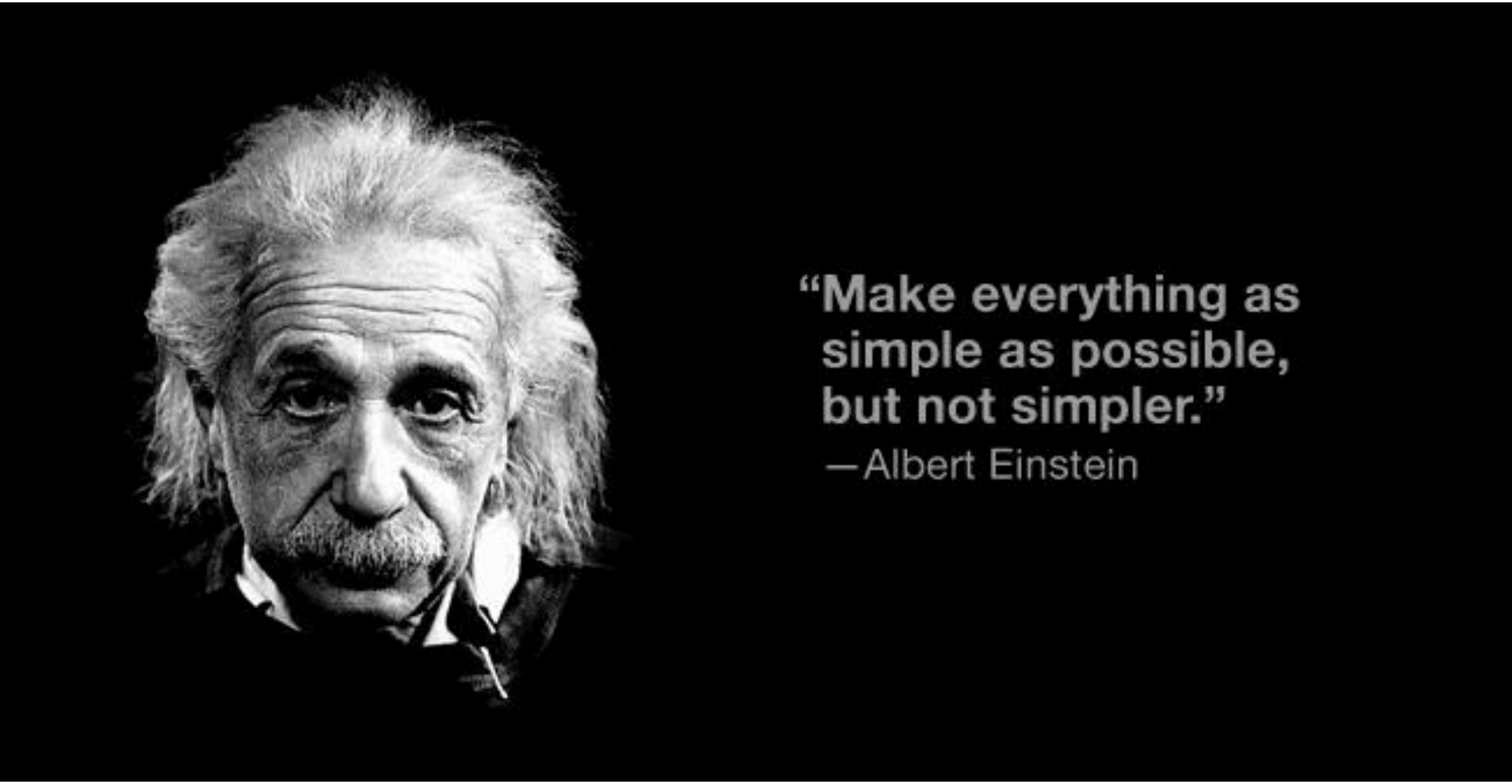
Asymptotic notations are a simplification of the “actual” runtime.

- ✧ It does not tell the whole story about how fast a program runs in reality.

  In real-world applications, constant factor matters! hardware matters! implementation matters!

- ✧ This simplification makes possible the development of the whole **theory of computational complexity**.

  IMPORTANT idea!


"Make everything as simple as possible, but not simpler."
—Albert Einstein

Quick note

In CSC165, we use **asymptotic notations** such as $\boldsymbol{O(n^2)}$, and sometimes, we use the **exact forms**, such as $\boldsymbol{3n^2 + 2n}$ to be more precise. It depends on the problem requirements.

ANALYZE THE TIME COMPLEXITY OF A PROGRAM

Linear Search

```
def LS(A, x):
””” Return index i, x == A[i].
    Otherwise, return -1 ”””
i = 0
while i < len(A):
    if A[i] == x:
        return i
    i = i + 1
return -1
```

What’s the runtime of this program?

Can’t say yet, it depends on the input (A, x).

Linear Search

```
def LS(A, x):
””” Return index i, x == A[i].
    Otherwise, return -1 ”””
i = 0
while i < len(A):
    if A[i] == x:
        return i
    i = i + 1
return -1
```

Count time complexity
(# of lines of code executed)

LS([2, 4, 6, 8], 4)

$t_{LS}([2, 4, 6, 8], 4) = 7$

Linear Search

```
def LS(A, x):
""" Return index i, x == A[i].
    Otherwise, return -1 """
    i = 0
    while i < len(A):
        if A[i] == x:
            return i
        i = i + 1
    return -1
```

Count time complexity
LS([2, 4, 6, 8], 6)

$t_{LS}([2, 4, 6, 8], 6) = 10$

Linear Search

```
def LS(A, x):
""" Return index i, x == A[i].
    Otherwise, return -1 """
i = 0
while i < len(A):
    if A[i] == x:
        return i
    i = i + 1
return -1
```

$t_{LS}([2, 4, 6, 8], 6) = 10$

What is the runtime of LS(A, x)?

if the first index where x is found is k

i.e., A[k] == x

$$t_{LS}(A, x) = 1 + 3(k+1)$$
$$= 3k + 4$$

Linear Search

```
def LS(A, x):
””” Return index i, x == A[i].
    Otherwise, return -1 ”””
i = 0
while i < len(A):
    if A[i] == x:
        return i
    i = i + 1
return -1
```

Count time complexity

LS([2, 4, 6, 8], 99)

$t_{LS}([2, 4, 6, 8], 99) = 15$

Linear Search

```
def LS(A, x):
""" Return index i, x == A[i].
    Otherwise, return -1 """
i = 0
while i < len(A):
    if A[i] == x:
        return i
    i = i + 1
return -1
```

$t_{LS}([2, 4, 6, 8], 99) = 15$

what is the runtime of LS(A, x)?

if x is not in A at all

let n be the size of A

$$t_{LS}(A, x) = 1 + 3n + 2$$

$$= 3n + 3$$

Takeaway

- program runtime varies with inputs
- among inputs of a given size, there is a **worst case** in which the runtime is the longest

Worst-case time Complexity

$t_P(x)$: running time of program P with input x

the worst-case time complexity of P with input $x \in I$ of size n

$$W_P(n) = \max\{\, t_P(x) \,|\, x \in I \wedge \mathrm{size}(x) = n \,\}$$

Linear Search

```
def LS(A, x):
""" Return index i, x == A[i].
    Otherwise, return -1 """
i = 0
while i < len(A):
    if A[i] == x:
        return i
    i = i + 1
return -1
```

What is the worst-case running time of LS(A, x), given that len(A) == n ?

$$W_{LS}(n) = 1 + 3n + 2$$
$$= 3n + 3$$

Worst-case: x is not in A at all!

$t_{LS}([2, 4, 6, 8], 99) = 15$

- **Worst-case**: performance in the worst situation, what we typically do in CSC165, and in CSC236

- **Best-case**: performance in the best situation, not very interesting, rarely studied

- **Average-case**: the expected performance under random inputs following certain probability distribution, will study in CSC263

Next class

- ➔ More on asymptotic notations & definitions
- ➔ Algorithm analysis