CSC165, Winter 2015
Assignment 3
Sample Solutions

IMPORTANT: You **must** use the proof structures and format of this course. Otherwise, you won't get full mark even if your answers are correct.

1. **Prove** or **disprove** each of the following claims.

   You may assume that $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$.

   (a) Let $f(n) = n\lfloor \frac{n}{2} \rfloor$, and $g(n) = n^2 - 2n + 1$.
   Then $f \in \Theta(g)$.
   Solution:

   Let $c_2 = 1$, $B_2 = 4$. Then $c_2 \in \mathbb{R}^+$ and $B_2 \in \mathbb{N}$.
   Assume n is a typical integer and $n \geq B_2$.
   Then $n^2 - 4n + 2 \geq 0$. # $n \geq B_2 \geq 4$
   Then $2n^2 - 4n + 2 \geq n^2$. # add n^2 to both sides
   Then $n^2 - 2n + 1 \geq n^2/2$. # divide both sides by 2
   Then $n^2 - 2n + 1 \geq n\lfloor \frac{n}{2} \rfloor$. # since $n^2/2 \geq n\lfloor \frac{n}{2} \rfloor$
   Then $c_2 g(n) \geq f(n)$. # $c_2 = 1$
   Then $\forall n \in \mathbb{N}, n \geq B_2 \Rightarrow f(n) \leq c_2 g(n)$. # introduce $\forall$ and $\Longrightarrow$
   Let $c_1 = 1/3$, $B_1 = 3$. Then $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$.
   Assume n is a typical integer and $n \geq B_1$.
   Then $n^2 + 4n \geq 14$. # $n \geq B_1 \geq 3$
   Then $3n^2 - 12 \geq 2n^2 - 4n + 2$. # add $2n^2 - 4n - 12$ to both sides
   Then $\frac{n^2}{2} - 2 \geq \frac{n^2-2n+1}{3}$. # divide both sides by 6
   Then $n\lfloor \frac{n}{2} \rfloor \geq \frac{n^2-2n+1}{3}$. # since $n\lfloor \frac{n}{2} \rfloor \geq \frac{n^2}{2} - 2$
   Then $c_1 g(n) \leq f(n)$. # $c_1 = 1/3$
   Then $\forall n \in \mathbb{N}, n \geq B_1 \Rightarrow c_1 g(n) \leq f(n)$. # introduce $\forall$ and $\Longrightarrow$
   Let $B = max(B_1, B_2)$.
   Then $\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$. # introduce $\exists$

   (b) Let $f(n) = n^4 + 3n^3 + n^2 - 1$, and $g(n) = n^5 - 8n^3 - n$.
   Then $f \in \Theta(g)$.
   Solution: f is not in $\Theta(g)$ since f is not in $\Omega(g)$.

   Assume $f \in \Omega(g)$. # to derive contradiction
   Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow \frac{f(n)}{g(n)} \geq c$. # by definition of Ω
   Also, $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{f(n)}{g(n)} < c$. # by definition of limits and since $lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
   Contradiction! # $\frac{f(n)}{g(n)}$ cannot be satisfy the above two statements at the same time

Then $f \notin \Omega(g)$. # assuming otherwise leads to contradiction

(c) Let $f(n) = n^n$, and $g(n) = n^{n-5}$.

Then $f \in \Theta(g)$.

Solution: f is not in $\Theta(g)$ since f is not in $\mathcal{O}(g)$.

Assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$. # arbitrary values

Then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies f(n) > c.g(n)$. # definition of $\lim_{n\to\infty} f(n)/g(n) = \infty$

Let n_1 be such that $\forall n \in \mathbb{N}, n \geq n_1 \implies f(n) > c.g(n)$. # instantiate n'

Let $n_0 = \max(B, n_1)$. Then $n_0 \in \mathbb{N}$.

Then $n_0 \geq B$. # by definition of max

Then $f(n_0) > cg(n_0)$. # by the assumption above $f(n) > cg(n)$, since $n_0 \geq n_1$

Then $n_0 \geq B \wedge f(n_0) \geq cg(n_0)$. # introduce $\wedge$

Then $\exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$. # introduce $\exists$

Then $\forall c \in \mathbb{R}, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge f(n) > cg(n)$. # introduce $\forall$

Then $f \notin \mathcal{O}(g)$. # the above statement is the negation of the definition of $\mathcal{O}$

2. Prove a **tight bound** on the worst-case running time of each of the following algorithms

(a)
```
def mystery1(L):
        """ L is a non-empty list of length len(L) = n. """
        if L[0] is even:
           i=0
           while i <n^2:
                L[0] = L[0] + L[i/n]
                i=i+1
        else:
           i=0
           while i < n-1:
                L[0] = L[0] - L[i]
                i=i+1
```

Solution: See the sample solutions to Tutorial 7.

(b)
```
def mystery2(L):
        """ L is a non-empty list of length len(L) = n. """
        step = 1
        index = 0
        while index < len(L):
           index = index + step
           step = step + 1
```

Solution:

Intuition: The number of iterations of the loop on any input L of length n will depend on the value of the variable index. And the value of index depends on the value of variable step. The variable step increases by 1 on each iteration of the loop. The variable step takes on the values:

$$step = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow ...$$

The variable index takes on the values:

$$index = 0 \rightarrow 0+1 \rightarrow 0+1+2 \rightarrow 0+1+2+3 \rightarrow ... \rightarrow 0+1+2+3+...+k \rightarrow ...$$

The value of index after the kth iteration of the loop is $0+1+2+3+...+k = k(k+1)/2$.

The number of iterations of the loop will be m, where m is such that $m(m+1)/2 > n$ (to get out of the loop) while $(m-1)((m-1)+1)/2 < n$. (To have a mth iteration of the loop, we need $(m-1)((m-1)+1)/2 < n$ or $(m-1)m/2 < n$) So we have $m(m-1) < 2n$ and $m(m+1) > 2n$. Hence, there will be $m \approx \lceil\sqrt{2n}\rceil$ iterations. (There wont be more than $\lceil\sqrt{2n}\rceil$ iterations. There wont be fewer than $\lfloor\sqrt{2n}\rfloor$ iterations.) We can justify this last conclusion more rigorously.

Consider the expression $m(m-1)$. We want $m(m-1) < 2n$. Suppose $m = \lceil\sqrt{2n}\rceil + 1$. Then, considering $m(m-1)$, we have

$$\begin{aligned}
(\lceil\sqrt{2n}\rceil + 1)((\lceil\sqrt{2n}\rceil + 1) - 1) &= (\lceil\sqrt{2n}\rceil + 1)(\lceil\sqrt{2n}\rceil) \\
&= (\lceil\sqrt{2n}\rceil)^2 + (\lceil\sqrt{2n}\rceil) \\
&\geq (\sqrt{2n})^2 + (\sqrt{2n}) \\
&= 2n + (\sqrt{2n}) \\
&> 2n
\end{aligned}$$

Hence, $m \leq \lceil \sqrt{2n} \rceil$ since otherwise $m(m-1) \nleq 2n$.
Similarly, we want $m(m+1) \geq 2n$. Suppose $m = \lfloor \sqrt{2n} \rfloor - 1$. Then

$$\begin{aligned}
(\lfloor \sqrt{2n} \rfloor - 1)((\lfloor \sqrt{2n} \rfloor - 1) + 1) &= (\lfloor \sqrt{2n} \rfloor - 1)(\lfloor \sqrt{2n} \rfloor) \\
&= (\lfloor \sqrt{2n} \rfloor)^2 - (\lfloor \sqrt{2n} \rfloor) \\
&\leq (\sqrt{2n})^2 - (\sqrt{2n}) \\
&= 2n - (\sqrt{2n}) \\
&< 2n
\end{aligned}$$

Hence, $m \geq \lfloor \sqrt{2n} \rfloor$, since otherwise $m(m+1) \ngeq 2n$.
Since the number of steps per iteration is constant, and the number of steps performed only depends on the length of the list L, we have that the worst-case runtime of the algorithm is $\Theta(\sqrt{n})$, where $n = len(L)$.
To prove this conclusion in a detailed way, we need to put the above arguments in a formal proof that follows the usual structure.
Let I be the set of possible inputs to the algorithm. That is, I is the set of nonempty lists of numbers.

Let $c_0 = 4\sqrt{2} + 7$ and $B_0 = 1$.
Then $c_0 \in \mathbb{R}^+$ and $B_0 \in \mathbb{N}$.
Assume $L \in I$ is an arbitrary list of length $n \geq B_0$

Then the algorithm executes 4 statements for each iteration of the loop (lines 4,5,6,7), and loops for at most $\lceil \sqrt{2n} \rceil$ iterations.
In addition, the algorithm has 2 initialization statements and the last evaluation of the loop condition.
Then,

$$\begin{aligned}
t(L) &\leq 4\lceil \sqrt{2n} \rceil + 3 \\
&\leq 4(\sqrt{2n} + 1) + 3 \\
&= 4\sqrt{2n} + 7 \\
&\leq 4\sqrt{2n} + 7\sqrt{n} \\
&\leq (4\sqrt{2} + 7)\sqrt{n} \\
&\leq c_0\sqrt{n}
\end{aligned}$$

Then $\forall L \in I, size(L) \geq B_0 \implies t(L) \leq c_0\sqrt{n}$. $\quad$ # $size(L) = n$
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall L \in I, size(L) \geq B \implies t(L) \leq c\sqrt{n}$.
Then $T(n) \in \mathcal{O}(\sqrt{n})$.
Let $c_0 = \sqrt{2}$ and $B_0 = 1$.
Then $c_0 \in \mathbb{R}^+$ and $B_0 \in \mathbb{N}$.
Assume $L \in I$ is an arbitrary list of length $n \geq B_0$

Let $L_0 = [1, 2, 3, ..., n]$. Then $L_0 \in I$ and $size(L_0) = n$. Then the algorithm executes 2 steps followed by at least $\lfloor \sqrt{2n} \rfloor$ iterations of at least one statement. Then,

$$\begin{aligned}
t(L_0) &\geq \lfloor \sqrt{2n} \rfloor + 2 \\
&> \sqrt{2n} \\
&= c_0\sqrt{n}
\end{aligned}$$

Then $\exists L \in I, size(L) = n \wedge t(L) \geq c_0\sqrt{n}$.

Then $\forall n \in \mathbb{N}, n \geq B_0 \implies \exists L \in I, size(L) = n \wedge t(L) \geq c_0\sqrt{n}$.
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \implies \exists L \in I, size(L) = n \wedge t(L) \geq c\sqrt{n}$.
Then $T(n) \in \Omega(\sqrt{n})$.
Then $T(n) \in \Theta(\sqrt{n})$.

3. Prove each of the following statements by induction (in all parts, assume that $n \in \mathbb{N}$)

(a) $1 + 6 + 11 + ... + (5n - 4) = \frac{n(5n-3)}{2}$, $n \geq 1$.

Solution: Let P(n) denotes $1 + 6 + 11 + ... + (5n - 4) = \frac{n(5n-3)}{2}$.

Prove $P(1)$**:** $(5n - 4) = 1$ and $\frac{n(5n-3)}{2} = \frac{1(5-3)}{2} = 1$. So $P(1)$ is true.
Prove $\forall n \in \mathbb{N}, P(n) \implies P(n + 1)$**:**
Assume $n \in \mathbb{N}$. # arbitrary natural number
Assume $P(n)$, that is $1 + 6 + 11 + 16 + ... + (5n - 4) = \frac{n(5n-3)}{2}$. # antecedent
Then $1 + 6 + 11 + ... + (5n - 4) + (5(n + 1) - 4) = \frac{n(5n-3)}{2} + (5(n + 1) - 4)$. # add $(5(n + 1) - 4)$ to both sides
Then $1 + 6 + 11 + ... + (5n - 4) + (5(n + 1) - 4) = \frac{5n^2-3n+10n+10-8}{2}$. # algebra
Then $1 + 6 + 11 + ... + (5n - 4) + (5(n + 1) - 4) = \frac{5n^2-7n+2}{2}$. # algebra
Then $1 + 6 + 11 + ... + (5n - 4) + (5(n + 1) - 4) = \frac{5n^2-7n+2}{2}$. # algebra
Also we have $(n + 1)(5(n + 1) - 3) = 5n^2 + 5n - 3n + 5n + 5 - 3$. # algebra
So $1+6+11+...+(5n-4)+(5(n+1)-4) = \frac{(n+1)(5(n+1)-3)}{2}$. # by the two previous lines
Then $P(n + 1)$. # by the previous line
Then $P(n) \implies P(n + 1)$. # introduce $\implies$
Then $\forall n \in \mathbb{N}, P(n) \implies P(n + 1)$. # introduce $\forall$

(b) For all natural numbers $n \geq 3$,
$4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n-16)}{3}$.

Solution: Let P(n) denotes $4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n-16)}{3}$.

Prove $P(3)$**:** $4^3 = 64$ and $\frac{4(4^3-16)}{3} = \frac{4(64-16)}{3} = 64$. So $P(3)$ is true.
Prove $\forall n \in \mathbb{N}, P(n) \implies P(n + 1)$**:**
Assume $n \in \mathbb{N}$, $n \geq 3$. # arbitrary natural number
Assume $P(n)$, that is $4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n-16)}{3}$. # antecedent
Then $4^3 + 4^4 + 4^5 + ... + 4^n + 4^{n+1} = \frac{4(4^n-16)}{3} + 4^{n+1}$. # add 4^{n+1} to both sides
Then $4^3 + 4^4 + 4^5 + ... + 4^n + 4^{n+1} = \frac{4^{n+1}+3*4^{n+1}-4*16)}{3}$. # algebra
Then $4^3 + 4^4 + 4^5 + ... + 4^n + 4^{n+1} = \frac{4(4^{n+1}-16)}{3}$. # algebra
Then $P(n + 1)$. # by the previous line
Then $P(n) \implies P(n + 1)$. # introduce $\implies$
Then $\forall n \in \mathbb{N}, P(n) \implies P(n + 1)$. # introduce $\forall$

(c) $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$, $n \geq 2$.
Solution:
First note that $(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = (\sqrt{n+1})^2 + (\sqrt{n})^2 = (n+1) - n = 1$. Therefore, $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, and so $\sqrt{n+1} = \sqrt{n} + \frac{1}{\sqrt{n+1}+\sqrt{n}}$.
Let P(n) denotes $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$.

Prove $P(2)$**:** $\sqrt{2} \approx 1.4$ and $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \approx 1.7$, so $P(2)$ is true.

Prove $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$**:**

Assume $n \in \mathbb{N}$, $n \geq 2$. # arbitrary natural number

Assume $P(n)$, that is $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$. # antecedent

Then $\sqrt{n} + \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}+\sqrt{n}}$. # add $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ to both sides

Then $\sqrt{n+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}+\sqrt{n}}$. # $\sqrt{n+1} = \sqrt{n} + \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Then $\sqrt{n+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$. # $\sqrt{n+1} < \sqrt{n+1} + \sqrt{n}$, so $\frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Then $P(n+1)$. # by the previous line

Then $P(n) \implies P(n+1)$. # introduce $\implies$

Then $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$. # introduce $\forall$