1. Prove or disprove each of the following claims.
   You may assume that $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ and $g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$.
   
   (a) Let $f(n) = n\lceil \frac{n}{2} \rceil$, and $g(n) = n^2 - 2n + 1$.
   Then $f \in \Theta(g)$.
   
   Solution: Let $c_1 = 1$, $B_1 = 3$. Then $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$.
   Assume $n$ is a typical integer and $n \geq B_1$.
   Then $n^2 + 4n \geq 14$. # $n \geq B_1 \geq 3$
   Then $3n^2 - 12 \geq 2n^2 - 4n + 2 \geq n^2 + 2n + 1 \geq n^2/2$. # divide both sides
   Then $n^2 - 2n + 1 \geq n^2/2$. # divide both sides by 2
   Then $n^2 - 2n + 1 \geq n\lceil \frac{n}{2} \rceil$. # since $n^2/2 \geq n\lceil \frac{n}{2} \rceil$
   Then $c_2 g(n) \geq f(n)$. # $c_2 = 1$
   Then $\forall n \in \mathbb{N}$, $n \geq B_2 \Rightarrow f(n) \leq c_2 g(n)$. # introduce $\forall$ and $\Rightarrow$
   Let $c_1 = 1/3$, $B_1 = 3$. Then $c_1 \in \mathbb{R}^+$ and $B_1 \in \mathbb{N}$.
   Assume $n$ is a typical integer and $n \geq B_1$.
   Then $n^2 + 4n \geq 14$. # $n \geq B_1 \geq 3$
   Then $3n^2 - 12 \geq 2n^2 - 4n + 2$. # add $2n^2 - 4n - 12$ to both sides
   Then $\frac{2n^2}{3} - 2 \geq \frac{2n^2 - 2n + 1}{3}$. # divide both sides by 6
   Then $n\lceil \frac{n}{2} \rceil \geq n\lceil \frac{n}{2} \rceil - \frac{n^2}{3} \geq \frac{n^2}{2} - 2$
   Then $c_1 g(n) \leq f(n)$. # $c_1 = 1/3$
   Then $\forall n \in \mathbb{N}$, $n \geq B_1 \Rightarrow c_1 g(n) \leq f(n)$. # introduce $\forall$ and $\Rightarrow$
   Let $B = \max(B_1, B_2)$.
   Then $\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}$, $n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$.
   
   (b) Let $f(n) = n^4 + 3n^3 + n^2 - 1$, and $g(n) = n^5 - 8n^3 - n$.
   Then $f \in \Theta(g)$.
   
   Solution: $f$ is not in $\Theta(g)$ since $f$ is not in $\Omega(g)$.
   Assume $f \in \Omega(g)$. # to derive contradiction
   Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}$, $n \geq B \Rightarrow \frac{f(n)}{g(n)} \geq c$. # by definition of $\Omega$
   Also, $\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}$, $n \geq n' \Rightarrow \frac{f(n)}{g(n)} < c$. # by definition of limits and since
   $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$
   Contradiction! # $\frac{f(n)}{g(n)}$ cannot be satisfy the above two statements at the same time
Then $f \notin \Omega(g)$.  # assuming otherwise leads to contradiction

(c) Let $f(n) = n^n$, and $g(n) = n^{n-5}$.

Then $f \in \Theta(g)$.

**Solution:** $f$ is not in $\Theta(g)$ since $f$ is not in $O(g)$.  
Assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$.  # arbitrary values

Then $\exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \implies f(n) > c.g(n)$.  # definition of $\lim_{n \to \infty} f(n)/g(n) = \infty$

Let $n_1$ be such that $\forall n \in \mathbb{N}, n \geq n_1 \implies f(n) > c.g(n)$.  # instantiate $n'$

Let $n_0 = \max(B, n_1)$.  Then $n_0 \in \mathbb{N}$.

Then $n_0 \geq B$.  # by definition of max

Then $f(n_0) > c.g(n_0)$.  # by the assumption above $f(n) > c.g(n)$, since $n_0 \geq n_1$

Then $n_0 \geq B \land f(n_0) \geq c.g(n_0)$.  # introduce $\land$

Then $\exists n \in \mathbb{N}, n \geq B \land f(n) > c.g(n)$.  # introduce $\exists$

Then $\forall c \in \mathbb{R}, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land f(n) > c.g(n)$.  # introduce $\forall$

Then $f \notin O(g)$.  # the above statement is the negation of the definition of $O$
2. Prove a **tight bound** on the worst-case running time of each of the following algorithms

(a) ```python
def mystery1(L):
    """ L is a non-empty list of length len(L) = n. """
    if L[0] is even:
        i=0
        while i < n^2:
            L[0] = L[0] + L[i/n]
            i=i+1
    else:
        i=0
        while i < n-1:
            L[0] = L[0] - L[i]
            i=i+1

Solution: See the sample solutions to Tutorial 7.
``` 

(b) ```python
def mystery2(L):
    """ L is a non-empty list of length len(L) = n. """
    step = 1
    index = 0
    while index < len(L):
        index = index + step
        step = step + 1

Solution:
Intuition: The number of iterations of the loop on any input $L$ of length $n$ will depend on the value of the variable index. And the value of index depends on the value of variable step. The variable step increases by 1 on each iteration of the loop. The variable step takes on the values:

\[
\text{step} = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow ...
\]

The variable index takes on the values:

\[
\text{index} = 0 \rightarrow 0+1 \rightarrow 0+1+2 \rightarrow 0+1+2+3 \rightarrow ... \rightarrow 0+1+2+3+...+k \rightarrow ...
\]

The value of index after the $k$th iteration of the loop is $0 + 1 + 2 + 3 + \ldots + k = k(k + 1)/2$.

The number of iterations of the loop will be $m$, where $m$ is such that $m(m + 1)/2 > n$ (to get out of the loop) while $(m - 1)((m - 1) + 1)/2 < n$. (To have a $m$th iteration of the loop, we need $(m - 1)((m - 1) + 1)/2 < n$ or $(m - 1)m/2 < n$) So we have $m(m - 1) < 2n$ and $m(m + 1) > 2n$. Hence, there will be $m \approx \lceil \sqrt{2n} \rceil$ iterations. (There wont be more than $\lceil \sqrt{2n} \rceil$ iterations.) We can justify this last conclusion more rigorously.

Consider the expression $m(m - 1)$. We want $m(m - 1) < 2n$. Suppose $m = \lceil \sqrt{2n} \rceil + 1$. Then, considering $m(m - 1)$, we have

\[
\begin{align*}
(\lceil \sqrt{2n} \rceil + 1)(\lceil \sqrt{2n} \rceil + 1) & = (\lceil \sqrt{2n} \rceil + 1)(\lceil \sqrt{2n} \rceil) \\
& = (\lceil \sqrt{2n} \rceil)^2 + (\lceil \sqrt{2n} \rceil) \\
& \geq (\sqrt{2n})^2 + (\sqrt{2n}) \\
& = 2n + (\sqrt{2n}) \\
& > 2n
\end{align*}
\]
Hence, \( m \leq \lceil \sqrt{2n} \rceil \) since otherwise \( m(m - 1) \neq 2n \).

Similarly, we want \( m(m + 1) \geq 2n \). Suppose \( m = \lfloor \sqrt{2n} \rfloor - 1 \). Then

\[
\begin{align*}
(\lfloor \sqrt{2n} \rfloor - 1)(\lfloor \sqrt{2n} \rfloor - 1 + 1) &= (\lfloor \sqrt{2n} \rfloor - 1)(\lfloor \sqrt{2n} \rfloor) \\
&= (\lfloor \sqrt{2n} \rfloor)^2 - (\lfloor \sqrt{2n} \rfloor) \\
&\leq (\sqrt{2n})^2 - (\sqrt{2n}) \\
&= 2n - (\sqrt{2n}) \\
&< 2n
\end{align*}
\]

Hence, \( m \geq \lfloor \sqrt{2n} \rfloor \), since otherwise \( m(m + 1) \neq 2n \).

Since the number of steps per iteration is constant, and the number of steps performed only depends on the length of the list \( L \), we have that the worst-case runtime of the algorithm is \( \Theta(\sqrt{n}) \), where \( n = \text{len}(L) \).

To prove this conclusion in a detailed way, we need to put the above arguments in a formal proof that follows the usual structure.

Let \( I \) be the set of possible inputs to the algorithm. That is, \( I \) is the set of nonempty lists of numbers.

Let \( c_0 = 4\sqrt{2} + 7 \) and \( B_0 = 1 \).

Then \( c_0 \in \mathbb{R}^+ \) and \( B_0 \in \mathbb{N} \).

Assume \( L \in I \) is an arbitrary list of length \( n \geq B_0 \).

Then the algorithm executes 4 statements for each iteration of the loop (lines 4, 5, 6, 7), and loops for at most \( \lceil \sqrt{2n} \rceil \) iterations.

In addition, the algorithm has 2 initialization statements and the last evaluation of the loop condition.

Then,

\[
t(L) \leq 4\lceil \sqrt{2n} \rceil + 3 \\
\leq 4(\sqrt{2n} + 1) + 3 \\
= 4\sqrt{2n} + 7 \\
\leq 4\sqrt{2n} + 7\sqrt{n} \\
\leq (4\sqrt{2} + 7)\sqrt{n} \\
\leq c_0\sqrt{n}
\]

Then \( \forall L \in I, \text{size}(L) \geq B_0 \implies t(L) \leq c_0\sqrt{n} \).  \# \( \text{size}(L) = n \)

Then \( \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall L \in I, \text{size}(L) \geq B \implies t(L) \leq c\sqrt{n} \).

Then \( T(n) \in \mathcal{O}(\sqrt{n}) \).

Let \( c_0 = \sqrt{2} \) and \( B_0 = 1 \).

Then \( c_0 \in \mathbb{R}^+ \) and \( B_0 \in \mathbb{N} \).

Assume \( L \in I \) is an arbitrary list of length \( n \geq B_0 \).

Let \( L_0 = [1, 2, 3, ..., \sqrt{2n}] \). Then \( L_0 \in I \) and \( \text{size}(L_0) = n \). Then the algorithm executes 2 steps followed by at least \( \lceil \sqrt{2n} \rceil \) iterations of at least one statement. Then,

\[
t(L_0) \geq \lceil \sqrt{2n} \rceil + 2 \\
> \sqrt{2n} \\
= c_0\sqrt{n}
\]

Then \( \exists L \in I, \text{size}(L) = n \land t(L) \geq c_0\sqrt{n} \).
3. Prove each of the following statements by induction (in all parts, assume that \( n \in \mathbb{N} \))

(a) \( 1 + 6 + 11 + ... + (5n - 4) = \frac{n(5n-3)}{2}, \ n \geq 1 \).

Solution: Let \( P(n) \) denotes \( 1 + 6 + 11 + ... + (5n - 4) = \frac{n(5n-3)}{2} \).

Prove \( P(1) \): \( 5(4) = 1 \) and \( \frac{5(4)-3}{2} = 1 \). So \( P(1) \) is true.

Prove \( \forall n \in \mathbb{N}, P(n) \implies P(n+1) \):
Assume \( n \in \mathbb{N} \). # arbitrary natural number

Assume \( P(n) \), that is \( 1 + 6 + 11 + ... + (5n - 4) = \frac{n(5n-3)}{2} \). # antecedent
Then \( 1 + 6 + 11 + ... + (5n - 4) + (5(n+1) - 4) = \frac{2n^2-3n+10n+10-8}{2} = n(5n-3) + 5(n+1) - 4 \). # add \( (5(n+1) - 4) \) to both sides
Then \( 1 + 6 + 11 + ... + (5n - 4) + (5(n+1) - 4) = \frac{5n^2-7n+2}{2} \). # algebra
Then \( 1 + 6 + 11 + ... + (5n - 4) + (5(n+1) - 4) = \frac{5n^2-7n+2}{2} \). # algebra
Also we have \( (n+1)(5(n+1) - 3) = 5n^2 + 5n - 3n + 5n + 5 - 3 \). # algebra
So \( 1 + 6 + 11 + ... + (5n - 4) + (5(n+1) - 4) = \frac{(n+1)(5(n+1)-3)}{2} \). # by the two previous lines
Then \( P(n+1) \). # by the previous line
Then \( P(n) \implies P(n+1) \). # introduce \( \implies \)

Then \( \forall n \in \mathbb{N}, P(n) \implies P(n+1) \). # introduce \( \forall \)

(b) For all natural numbers \( n \geq 3 \),
\( 4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n-16)}{3} \).

Solution: Let \( P(n) \) denotes \( 4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n-16)}{3} \).

Prove \( P(3) \): \( 4^3 = 64 \) and \( \frac{4(4^3-16)}{3} = \frac{4(64-16)}{3} = 64 \). So \( P(3) \) is true.

Prove \( \forall n \in \mathbb{N}, P(n) \implies P(n+1) \):
Assume \( n \in \mathbb{N}, n \geq 3 \). # arbitrary natural number

Assume \( P(n) \), that is \( 4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n-16)}{3} \). # antecedent
Then \( 4^3 + 4^4 + 4^5 + ... + 4^n + 4^{n+1} = \frac{4(4^n-16)}{3} + 4^{n+1} \). # add \( 4^{n+1} \) to both sides
Then \( 4^3 + 4^4 + 4^5 + ... + 4^n + 4^{n+1} = \frac{4^{n+1}-4^3}{3} + 4^{n+1} \). # algebra
Then \( 4^3 + 4^4 + 4^5 + ... + 4^n + 4^{n+1} = \frac{4(4^{n+1}-16)}{3} \). # algebra
Then \( P(n+1) \). # by the previous line
Then \( P(n) \implies P(n+1) \). # introduce \( \implies \)

Then \( \forall n \in \mathbb{N}, P(n) \implies P(n+1) \). # introduce \( \forall \)

(c) \( \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}, \ n \geq 2 \).

Solution:
First note that \((\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = (\sqrt{n+1})^2 + (\sqrt{n})^2 = (n+1) - n = 1 \). Therefore, \( \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \), and so \( \sqrt{n+1} = \sqrt{n} + \frac{1}{\sqrt{n+1}+\sqrt{n}} \).

Let \( P(n) \) denotes \( \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} \).
Prove $P(2)$: $\sqrt{2} \approx 1.4$ and $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \approx 1.7$, so $P(2)$ is true.

Prove $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$:

Assume $n \in \mathbb{N}$, $n \geq 2$. # arbitrary natural number

Assume $P(n)$, that is $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$. # antecedent

Then $\sqrt{n} + \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1} + \sqrt{n}}$. # add $\frac{1}{\sqrt{n+1} + \sqrt{n}}$ to both sides

Then $\sqrt{n+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1} + \sqrt{n}}$. # $\sqrt{n+1} = \sqrt{n} + \frac{1}{\sqrt{n+1} + \sqrt{n}}$

Then $\sqrt{n+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1} + \sqrt{n}}$. # $\sqrt{n+1} < \sqrt{n+1} + \sqrt{n}$, so

$\sqrt{n+1} > \frac{1}{\sqrt{n+1} + \sqrt{n}}$

Then $P(n+1)$. # by the previous line

Then $P(n) \implies P(n+1)$. # introduce $\implies$

Then $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$. # introduce $\forall$