CSC165, Winter 2015
Assignment 2
Sample Solutions

IMPORTANT: You **must** use the proof structures and format of this course. Otherwise, you won't get full mark even if your answers are correct.

1. **Prove** or **disprove** each of the following claims.

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$.

   Solution: The claim is false. I will disprove it by proving the negation of the claim which is the following statement:

   $$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \lceil x+y \rceil \neq \lceil x \rceil + \lceil y \rceil.$$

   proof:

   Let $x = 1.2, y = 1.2$. Then $x, y \in \mathbb{R}$. # since $1.2 \in \mathbb{R}$
   Then $\lceil x+y \rceil = 3$. # by definition of the ceiling function since $x + y = 2.4$
   Then $\lceil x \rceil + \lceil y \rceil = 4$. # by definition of the ceiling function since $\lceil x \rceil = \lceil y \rceil = 2$
   Then $\lceil x+y \rceil \neq \lceil x \rceil + \lceil y \rceil$. # $3 \neq 4$
   Then $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \lceil x+y \rceil \neq \lceil x \rceil + \lceil y \rceil$. # introduced $\exists$

(b) For all integers x, y, and z, if $x \nmid y.z$ then $x \nmid y$ and $x \nmid z$. (Note that the symbol $\nmid$ denotes "does not divide")

Solution: The claim is true. Here's the translation of the claim in logical form:

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, (x \nmid y.z) \Rightarrow (x \nmid y) \wedge (x \nmid z).$$

proof:

Assume $x, y, z \in \mathbb{Z}$. # x, y, z are typical integers
  Assume $(x \mid y) \vee (x \mid z)$. # antecedent of contrapositive
    Case 1: Assume $x \mid y$.
      Then $\exists k_0 \in \mathbb{Z}$ such that $y = k_0.x$ # definition of $\mid$
      Then $y.z = k_0.x.z$ # multiple both sides by z
      Then $\exists k \in \mathbb{Z}$ such that $y.z = k.x$ # $k = k_0.z$, $k \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under $\times$
      Then $x \mid y.z$ # definition of $\mid$
    Case 2: Assume $x \mid z$. # antecedent
      Then $\exists k_0 \in \mathbb{Z}$ such that $z = k_0.x$ # definition of $\mid$
      Then $y.z = k_0.x.z$ # multiple both sides by y
      Then $\exists k \in \mathbb{Z}$ such that $y.z = k.x$ # $k = k_0.y$, $k \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under $\times$
      Then $x \mid y.z$ # definition of $\mid$
    Then $x \mid y.z$ # true for both cases
  Then $(x \mid y) \vee (x \mid z) \Rightarrow (x \mid y.z)$ # introduced $\Rightarrow$
  Then $(x \nmid y.z) \Rightarrow (x \nmid y) \wedge (x \nmid z)$ # implication is equivalent to contrapositive
Then $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, (x \nmid y.z) \Rightarrow (x \nmid y) \wedge (x \nmid z)$ # introduced $\forall$

2. Use **proof by contradiction** to prove that for all prime numbers x, y, and z, $x^2 + y^2 \neq z^2$.

   Solution: As the first step, we should translate the claim into logical form:

   Let P denotes the set of all prime numbers.

$$\forall x \in P, \forall y \in P, \forall z \in P, (x^2 + y^2 \neq z^2).$$

   To derive a contradiction, I assume the negation of the claim:

$$\exists x \in P, \exists y \in P, \exists z \in P, (x^2 + y^2 = z^2).$$

   proof:

   Assume $\exists x \in P, \exists y \in P, \exists z \in P, (x^2 + y^2 = z^2)$. # to derive contradiction
   - Let $x_0, y_0, z_0 \in P$ such that $x_0^2 + y_0^2 = z_0^2$ # instantiate $\exists$
   - Then $x_0^2 = z_0^2 - y_0^2 = (z_0 - y_0)(z_0 + y_0)$. # algebra
   - Also factors of x^2 are $1, x, x^2$. # x is a prime number
   - And $(z_0 + y_0) \neq x$ and $(z_0 - y_0) \neq x$. # $(z_0 + y_0) \neq (z_0 - y_0)$ as y_0 and z_0 are primes, and so are > 0
   - Then $((z_0 - y_0 = 1) \wedge (z_0 + y_0 = x_0^2)) \vee ((z_0 + y_0 = 1) \wedge (z_0 - y_0 = x_0^2))$. # only possible cases
   - Case 1: Assume $(z_0 - y_0 = 1) \wedge (z_0 + y_0 = x_0^2)$.
     - Then z_0 is the successor of y_0. # since $z_0 - y_0 = 1$
     - Then $z_0 = 3$ and $y_0 = 2$. # 2 and 3 are the only successive primes
     - Then $z_0 + y_0 = 5$. # $z_0 = 3$ and $y_0 = 2$
     - Contradiction! # $z_0 + y_0 = x_0^2 = 5$ and $x_0 \in \mathbb{N}$, but 5 is not square of any natural number
   - Case 2: Assume $(z_0 + y_0 = 1) \wedge (z_0 - y_0 = x_0^2)$.
     - Also $y_0 + z_0 > 1$. # y_0 and z_0 are primes, and so $y_0 > 1, z_0 > 1$
     - Contradiction! # by assumption $z_0 + y_0 = 1$
   - Contradiction! # derived contradiction for both cases

   Then $\forall x \in P, \forall y \in P, \forall z \in P, (x^2 + y^2 \neq z^2)$. # assuming the negation leads to a contradiction

3. Consider the definition of the floor function:

$$\mathbf{Def_1} : \forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, (y = \lfloor x \rfloor) \Leftrightarrow (y \le x) \wedge (\forall z \in \mathbb{Z}, (z \le x) \Rightarrow (z \le y)).$$

Use the proof structures of this course and $\mathbf{Def_1}$ to prove the following claims

(a) $\mathbf{S_1} : \forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \le y) \wedge (y < 1) \Rightarrow (\lfloor n + y \rfloor = n)$.
Note: In your proof, you may ONLY use those properties of the floor function that are specified by $\mathbf{Def_1}$.
Solution:

Assume $y \in \mathbb{R}$, $n \in \mathbb{Z}$. # y is a typical real number, n is a typical integer
  Assume $(0 \le y) \wedge (y < 1)$. # antecedent
    Then $(n \le n + y)$ and $(n + y < n + 1)$. # add n to both sides of the inequalities
    Assume $z \in \mathbb{Z}$. # z is a typical integer
      Assume $z \le n + y$. # antecedent
        Then $z < n + 1$. # $n + y < n + 1$ and transitivity of $<$
        Then $z \le n$. # $z < n + 1$, $n + 1$ is the successor of n and there is no integer between two successor integers
      Then $(z \le n + y) \Rightarrow (z \le n)$. # introduced $\Rightarrow$
    Then $\forall z \in \mathbb{Z}, (z \le n + y) \Rightarrow (z \le n)$. # introduced $\forall$
    Then $(n \le n + y) \wedge \forall z \in \mathbb{Z}, (z \le n + y) \Rightarrow (z \le n)$. # introduced $\wedge$
    Then $\lfloor n + y \rfloor = n$. # by $\mathbf{Def_1}$
  Then $(0 \le y) \wedge (y < 1) \Rightarrow (\lfloor n + y \rfloor = n)$. # introduced $\Rightarrow$
Then $\forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \le y) \wedge (y < 1) \Rightarrow (\lfloor n + y \rfloor = n)$. # introduced $\forall$

(b) $\mathbf{S_2} : \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (0 \le y) \wedge (y < 1) \wedge (x = \lfloor x \rfloor + y)$.
Note: In your proof, you may ONLY use those properties of the floor function that are specified by $\mathbf{Def_1}$.
Solution: First, I prove the following lemma:

$$\mathbf{Lemma1} : \forall x \in \mathbb{R}, (x - \lfloor x \rfloor < 1).$$

Proof for Lemma1:

Assume $x \in \mathbb{R}$. # x is a typical real number
  Then $\lfloor x \rfloor < \lfloor x \rfloor + 1$. # add $\lfloor x \rfloor$ to both sides of $0 < 1$
  Then $x < \lfloor x \rfloor + 1$. # by contrapositive in $\mathbf{Def_1}$ since $\lfloor x \rfloor + 1 \in \mathbb{Z}$
  Then $x - \lfloor x \rfloor < 1$. # deduct $\lfloor x \rfloor$ from both sides
Then $\forall x \in \mathbb{R}, (x - \lfloor x \rfloor < 1)$. # introduced $\forall$

Proof for $\mathbf{S_2}$:

Assume $x \in \mathbb{R}$. # x is a typical real number

Let $y = x - \lfloor x \rfloor$. Then $y \in \mathbb{R}$. # $x, \lfloor x \rfloor \in \mathbb{R}$ and $\mathbb{R}$ is closed under $-$

Then $x = \lfloor x \rfloor + y$. # add $\lfloor x \rfloor$ to both sides of $y = x - \lfloor x \rfloor$

Then $0 \leq y$. # by $\mathbf{Def_1}$, $x \geq \lfloor x \rfloor$, so $y = x - \lfloor x \rfloor \geq 0$

Then $y < 1$. # by **Lemma1** and $y = x - \lfloor x \rfloor < 1$

Then $(0 \leq y) \wedge (y < 1) \wedge (x = \lfloor x \rfloor + y)$. # introduced $\wedge$

Then $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (0 \leq y) \wedge (y < 1) \wedge (x = \lfloor x \rfloor + y)$. # introduced $\forall$

(c) $\mathbf{S_3} : \forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (\lfloor x + n \rfloor = \lfloor x \rfloor + n)$.

Note: In your proof, you may ONLY use those properties of the floor function that are specified by $\mathbf{Def_1}$, $\mathbf{S_1}$, and $\mathbf{S_2}$.

Solution:

Assume $x \in \mathbb{R}$, $n \in \mathbb{Z}$. # x is a typical real number, n is a typical integer

Then $\exists y \in \mathbb{R}$ such that $(0 \leq y < 1)$ and $(x = \lfloor x \rfloor + y)$. # by $\mathbf{S_2}$

Then $x + n = \lfloor x \rfloor + n + y$. # add n to both sides of $x = \lfloor x \rfloor + y$

Then $\lfloor x + n \rfloor = \lfloor \lfloor x \rfloor + n + y \rfloor = \lfloor x \rfloor + n$. # by $\mathbf{S_1}$ since $(0 \leq y < 1)$ and $\lfloor x \rfloor, n \in \mathbb{Z}$ and so $\lfloor x \rfloor + n \in \mathbb{Z}$

Then $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (\lfloor x + n \rfloor = \lfloor x \rfloor + n)$. # introduced $\forall$