CSC165, Winter 2015
Assignment 2
Sample Solutions

IMPORTANT: You **must** use the proof structures and format of this course. Otherwise, you won't get full mark even if your answers are correct.

1. **Prove** or **disprove** each of the following claims.

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.
   Solution: The claim is false. I will disprove it by proving the negation of the claim which is the following statement:
   $$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil.$$

   proof:
   Let $x = 1.2, y = 1.2$. then $x, y \in \mathbb{R}$. # since $1.2 \in \mathbb{R}$
   then $\lceil x + y \rceil = 3$. # by definition of the ceiling function since $x + y = 2.4$
   then $\lceil x \rceil + \lceil y \rceil = 4$. # by definition of the ceiling function since $\lceil x \rceil = \lceil y \rceil = 2$
   then $\lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil$. # $3 \neq 4$
   then $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil$. # introduced $\exists$

(b) For all integers x, y, and z, if $x \nmid y.z$ then $x \nmid y$ and $x \nmid z$. (Note that the symbol $\nmid$ denotes "does not divide")

Solution: The claim is true. Here's the translation of the claim in logical form:

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, (x \nmid y.z) \Rightarrow (x \nmid y) \wedge (x \nmid z).$$

Proof: **# prove by contrapositive**

Assume $x, y, z \in \mathbb{Z}$. # x, y, z are typical integers

assume $(x \mid y) \vee (x \mid z)$ # antecedent of contrapositive

Case 1: Assume $x \mid y$ # antecedent

then $\exists k_1 \in \mathbb{Z}$ such that $y = k_1 * x$ # definition of $\mid$

then $y * z = k_1 * x * z$ # multiple both sides by z

then $\exists k \in \mathbb{Z}$ such that $y * z = k * x$ # $let : k = k_1 * z$, $k \in \mathbb{Z}$, closed under $\times$

then $x \mid y * z$ # definition of $\mid$

Case 2: Assume $x \mid z$ # antecedent

then $\exists k_2 \in \mathbb{Z}$ such that $z = k_2 * x$ # definition of $\mid$

then $y * z = k_2 * x * z$ # multiple both sides by y

then $\exists k \in \mathbb{Z}$ such that $y * z = k * x$ # $let : k = k_2.y$, $k \in \mathbb{Z}$, closed under $\times$

then $x \mid y * z$ # definition of $\mid$

then $x \mid y * z$ # true for both cases

then $(x \mid y) \vee (x \mid z) \Rightarrow (x \mid y * z)$ # introduced $\Rightarrow$

then $(x \nmid y * z) \Rightarrow (x \nmid y) \wedge (x \nmid z)$ # implication is equivalent to its contrapositive

then $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, (x \nmid y * z) \Rightarrow (x \nmid y) \wedge (x \nmid z)$ # introduced $\forall$

2. Use **proof by contradiction** to prove that for all prime numbers x, y, and z, $x^2 + y^2 \neq z^2$.

Solution: Here's the translation of the claim in logical form:
Let P: set of all prime numbers, prove SP:

$$\forall x \in P, \forall y \in P, \forall z \in P, (x^2 + y^2 \neq z^2).$$

Negation of SP:

$$\exists x \in P, \exists y \in P, \exists z \in P, (x^2 + y^2 = z^2).$$

Proof: **# prove by contradiction**

Assume $\exists x \in P, \exists y \in P, \exists z \in P, (x^2 + y^2 = z^2)$. # derive contradiction $\neg$SP
  Let $x_0, y_0, z_0 \in P$ such that $x_0^2 + y_0^2 = z_0^2$ # instantiate $\exists$
  then $x_0^2 = z_0^2 - y_0^2 = (z_0 - y_0)(z_0 + y_0)$. # algebra
  then factors of x^2 are $1, x, x^2$. # x is a prime number
  and $(z_0 + y_0) \neq (z_0 - y_0)$. # y_0 and z_0 are primes, $y_0 \neq 0$
  then $((z_0 - y_0 = 1) \wedge (z_0 + y_0 = x_0^2)) \vee ((z_0 + y_0 = 1) \wedge (z_0 - y_0 = x_0^2))$
  # $(z_0 - y_0)$ and $(z_0 + y_0)$ can only be 1 or x_0^2, as both are interger factors.
  Case 1: Assume $(z_0 - y_0 = 1) \wedge (z_0 + y_0 = x_0^2)$.
    then z_0 is the successor of y_0. # $z_0 - y_0 = 1$
    then $z_0 = 3$ and $y_0 = 2$. # 2 and 3 are the only successive primes, P = {2,3,5, ...}
    then $z_0 + y_0 = 5$. # $z_0 = 3$ and $y_0 = 2$
    Contradiction! # $z_0 + y_0 = x_0^2 = 5$ but 5 is not square of any interger, contradict to $x_0 \in P$
  Case 2: Assume $(z_0 + y_0 = 1) \wedge (z_0 - y_0 = x_0^2)$.
    then $y_0 + z_0 \geq 4$. # y_0 and z_0 are primes, and so $y_0 \geq 2, z_0 \geq 2$
    Contradiction! # $y_0 + z_0 \geq 4$ contradict to $z_0 + y_0 = 1$
  Contradiction! # contradiction in both cases
then $\forall x \in P, \forall y \in P, \forall z \in P, (x^2 + y^2 \neq z^2)$ # negation of the statement ($\neg$SP) is false, then SP

3. Consider the definition:

$$\mathbf{Def_1}: \forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, (y = \lfloor x \rfloor) \Leftrightarrow (y \leq x) \wedge (\forall z \in \mathbb{Z}, (z \leq x) \Rightarrow (z \leq y)).$$

Use the proof structures of this course and $\mathbf{Def_1}$ to prove the following claims

(a) $\mathbf{S_1}: \forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \leq y) \wedge (y < 1) \Rightarrow (\lfloor n + y \rfloor = n)$.

Note: In your proof, you may ONLY use those properties of the floor function that are specified by $\mathbf{Def_1}$.

Proof: **# direct proof**

Assume $n \in \mathbb{Z}$, $y \in \mathbb{R}$ # y is a typical real number, n is a typical integer
  assume $(0 \leq y) \wedge (y < 1)$. # antecedent
    assume $z \in \mathbb{Z}$. # z is a typical integer
      assume $z \leq n + y$. # antecedent
        then $(n \leq n + y)$ # $0 \leq y$, add n to both sides of the inequalities
        then $(n + y < n + 1)$ # $y < 1$,add n to both sides of the inequalities
        $z < n + 1$. # $n + y < n + 1$ and transitivity of $<$
        then $z \leq n$ # $n < n + 1$ and $z \in \mathbb{Z}$, and there is no integer between two successor integers
      then $then(z \leq n + y) \Rightarrow (z \leq n)$. # introduced $\Rightarrow$
    then $\forall z \in \mathbb{Z}, (z \leq n + y) \Rightarrow (z \leq n)$. # introduced $\forall$
    then $(n \leq n + y) \wedge (\forall z \in \mathbb{Z}, (z \leq n + y) \Rightarrow (z \leq n))$. # introduced $\wedge$
    then $\lfloor n + y \rfloor = n$. # by $\mathbf{Def_1}$
  then $(0 \leq y) \wedge (y < 1) \Rightarrow (\lfloor n + y \rfloor = n)$. # introduced $\Rightarrow$
then $\forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \leq y) \wedge (y < 1) \Rightarrow (\lfloor n + y \rfloor = n)$. # introduced $\forall$