CSC165, Winter 2015
Assignment 1
Sample Solutions

1. Let $B(x)$ denotes "x plays this 2-person board game" and $P(x, y)$ denotes "x has played this 2-person board with y", where the domain for variables x and y consists of all students in your class.

   Questions:

   - Express each of the following sentences in logical notation. Define all sets and predicate symbols that you use in the logical expressions.
   - Write the negation of each of the sentences in English and in logical form. Simplifying the logical sentences so that only predicates are negated.

   Solution: Let C denotes the set of all students in your class.

   (a) [**4 Marks**] If Carol and Dan play this 2-person board game, then they have played it with each other.

   (**Hint**: note that $P(x, y)$ only expresses that x has played with y; this does **not** necessarily mean that y has played with x).

   $$B(Carol) \wedge B(Dan) \Rightarrow P(Carol, Dan) \wedge P(Dan, Carol).$$

   Negation: Carol and Dan play this 2-person board game, but they have not played it with each other.

   $$B(Carol) \wedge B(Dan) \wedge (\neg P(Carol, Dan) \vee \neg P(Dan, Carol)).$$

   (b) [**3 Marks**] No one in the class has played this 2-person board game with Bob.

   $$\forall x \in C, \neg P(x, Bob).$$

   Negation: Someone in the class has played this 2-person board game with Bob.

   $$\exists x \in C, P(x, Bob).$$

   (c) [**3 Marks**] Someone in your class does not play this 2-person board game.

   $$\exists x \in C, \neg B(x).$$

   Negation: All people in your class play this 2-person board game.

   $$\forall x \in C, B(x).$$

(d) [**5 Marks**] Everyone in your class who plays this 2-person board game has played the game with at least one other student in your class.

$$\forall x \in C, B(x) \Rightarrow (\exists y \in C, (x \neq y) \wedge P(x, y)).$$

Negation: Someone in your class plays this 2-person board game, but has not played it with other student in the class.

$$\exists x \in C, B(x) \wedge (\forall y \in C, (x = y) \vee \neg P(x, y)).$$

(e) [**4 Marks**] A student in your class has played this 2-person board game with everyone in your class.

$$\exists x \in C, \forall y \in C, P(x, y).$$

Negation: No one in your class has played this 2-person board game with everyone in the class.

$$\forall x \in C, \exists y \in C, \neg P(x, y).$$

2. Let S denotes the set of all students in your class, C denotes the set of all countries, and $T(x, y)$ denotes that x travels to country y.

   Express each of the following statements by a simple English sentence.

   Avoid symbols (e.g. x) and predicates (e.g. T(x,y)) in English sentences.

   (a) [**1 Mark**] $\neg T(Bob, y)$.
   Solution: Bob does not travel to **the** country.

   (b) [**1 Mark**] $\exists x \in S, T(x, Denmark) \wedge \forall x \in S, T(x, France)$.
   Solution: Some students in your class travel to Denmark and all students in the class travel to France.

   (c) [**1 Mark**] $\exists y \in C, T(Mona, y) \vee T(Pete, y)$.
   Solution: At least one of Mona or Pete travels to some country.

   (d) [**2 Marks**] $\forall x \in S, \forall z \in S, \exists y \in C, (x \neq z) \Rightarrow \neg(T(x, y) \wedge T(z, y))$.
   Solution: For every two different students in your class, exists a country such that at most one of the two students travel to.

   (e) [**2 Marks**] $\exists x \in S, \exists z \in S, \forall y \in C, (x \neq z) \wedge (T(x, y) \Leftrightarrow T(z, y))$.
   Solution: At least two different students in your class travel to the exact same countries.

   (f) [**2 Marks**] $\forall x \in S, \forall z \in S, \exists y \in C, (T(x, y) \Leftrightarrow T(z, y))$.
   Solution: For every two students in your class, there exists a country that either they both travel to, or none of them travel to.

3. Give the contrapositive and converse of the following statements in English.

   (Hint: it might be helpful if you first write these sentences in logical notation)

   (a) [**2 Marks**] If Socrates is a man and all men are mortal, then Socrates is mortal.
   Contrapositive: If Socrates is not mortal, then he is not a man or exists a man who is not mortal.
   Converse: If Socrates is mortal, then he is a man and all men are mortal.

(b) [**2 Marks**] If it does not rain or it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on.

Contrapositive: If the sailing race won't be held or the lifesaving demonstration won't go on, then it rains and it is foggy.

Converse: If the sailing race will be held and the lifesaving demonstration will go on, then it does not rain or it is not foggy.

(c) [**2 Marks**] For me, playing hockey is sufficient for being sore the next day.

Contrapositive: If I am not sore the next day, then I didn't play hockey.

Converse: If I am sore the next day, then I played hockey.

4. Suppose T is a set of natural numbers, and we have the following statement about T.

$\mathbf{S_1}$: 2 is the only prime number that divides elements of T.

Which of the following statements imply $\mathbf{S_1}$? Which are implied by $\mathbf{S_1}$? For each case, give a brief justification.

(a) [**6 Marks**] $\mathbf{S_2}$: T is the set of all natural numbers that satisfy the quadratic equation $x^2+x+1=0$.

$\mathbf{S_2}$ implies $\mathbf{S_1}$ because $\mathbf{S_2}$ is an empty set because the implication is vacuously true. In other words, all elements in the empty set satisfy $\mathbf{S_1}$.

$\mathbf{S_1}$ does not imply $\mathbf{S_2}$ because there are non-empty sets that satisfy $\mathbf{S_1}$.

(b) [**6 Marks**] $\mathbf{S_3}$: $T = \{16, 8, 528\}$.

$\mathbf{S_1}$ and $\mathbf{S_3}$ do not imply each other: 528 is dividable by 3 and 11, so whenever $\mathbf{S_1}$ is True $\mathbf{S_3}$ is False and whenever $\mathbf{S_3}$ is True $\mathbf{S_1}$ is False.

(c) [**6 Marks**] $\mathbf{S_4}$: All elements of T are equal to 2^n, where n is a natural number.

$\mathbf{S_1}$ implies $\mathbf{S_4}$. If the only prime factor of elements in T is 2, then every element of T is a power of 2. However, $2^0 = 1$ is not dividable by 2, so $\mathbf{S_4}$ does not imply $\mathbf{S_1}$.

(d) [**6 Marks**] $\mathbf{S_5}$: If $i, j \in T$, and $i < j$, then i divides j.

$\mathbf{S_1}$ implies $\mathbf{S_5}$, since if $\mathbf{S_1}$ is true, then smaller elements of T divide larger elements. However, $\mathbf{S_5}$ doesn't imply $\mathbf{S_1}$, since $\{3, 6, 12\}$ satisfies $\mathbf{S_5}$ but not $\mathbf{S_1}$.

5. Prove that the following equivalences are tautology. Show your work in detail. You **may not** use truth tables or Venn diagrams in your proof, but you are allowed to use manipulation rules (presented in Section 2.17 of the course notes).

For each step, you **must** clearly state which rule you apply.

(a) [**6 Marks**] $((P \Rightarrow Q_1) \wedge (P \Rightarrow Q_2)) \Leftrightarrow (P \Rightarrow (Q_1 \wedge Q_2))$

Solution:

$$\begin{aligned} (P \Rightarrow (Q_1 \wedge Q_2)) &\Leftrightarrow (\neg P \vee (Q_1 \wedge Q_2)) && \text{implication} \\ &\Leftrightarrow ((\neg P \vee Q_1) \wedge (\neg P \vee Q_2)) && \text{distribute } \vee \text{ over } \wedge \\ &\Leftrightarrow ((P \Rightarrow Q_1) \wedge (P \Rightarrow Q_2)) && \text{implication} \end{aligned}$$

(b) [**16 Marks**] $((\exists x \in D, P(x) \vee Q(x)) \Rightarrow R(y)) \Leftrightarrow (((\exists x \in D, P(x)) \Rightarrow R(y)) \wedge ((\exists x \in D, Q(x)) \Rightarrow R(y)))$

Solution:

$$
\begin{aligned}
((\exists x \in D, P(x) \vee Q(x)) \Rightarrow R(y)) &\Leftrightarrow (\neg (\exists x \in D, P(x) \vee Q(x)) \vee R(y)) && \text{implication} \\
&\Leftrightarrow ((\forall x \in D, \neg (P(x) \vee Q(x))) \vee R(y)) && \text{quantifier negation} \\
&\Leftrightarrow ((\forall x \in D, \neg P(x) \wedge \neg Q(x)) \vee R(y)) && \text{De Morgan's} \\
&\Leftrightarrow (((\forall x \in D, \neg P(x)) \wedge (\forall x \in D, \neg Q(x))) \vee R(y)) && \text{quantifier distributive} \\
&\Leftrightarrow (((\forall x \in D, \neg P(x)) \vee R(y)) \wedge ((\forall x \in D, \neg Q(x)) \vee R(y))) && \text{distribute } \vee \text{ over } \wedge \\
&\Leftrightarrow (\neg (\forall x \in D, \neg P(x)) \Rightarrow R(y)) \wedge (\neg (\forall x \in D, \neg Q(x)) \Rightarrow R(y)) && \text{implication} \\
&\Leftrightarrow (((\exists x \in D, P(x)) \Rightarrow R(y)) \wedge ((\exists x \in D, Q(x)) \Rightarrow R(y))) && \text{quantifier negation}
\end{aligned}
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